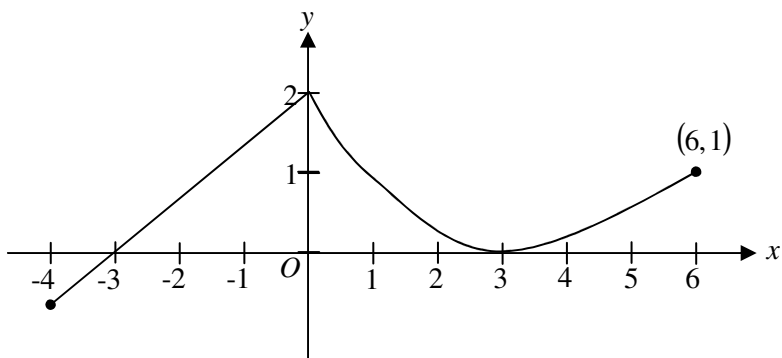


## PROBLEM SHEET S2 – AP Review

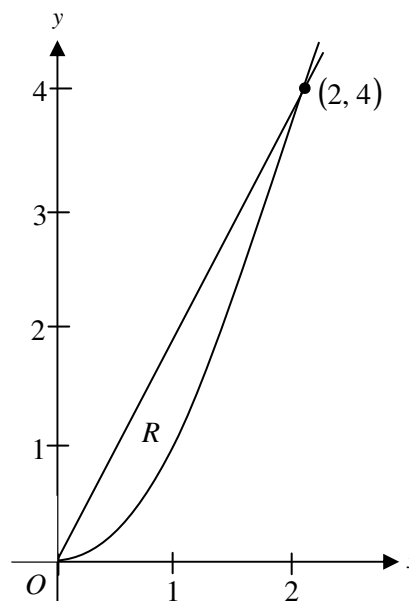
Questions marked with a ♦ allow use of a calculator.

Graph of  $f$ 

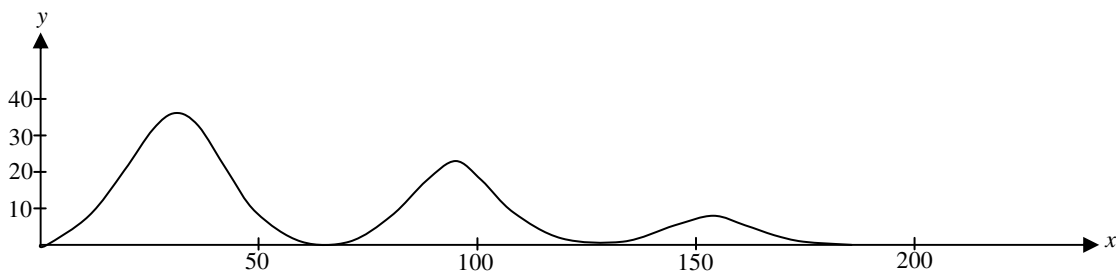
1. ♦ A continuous function  $f$  is defined on the closed interval  $-4 \leq x \leq 6$ . The graph of  $f$  consists of a line segment and a curve that is tangent to the  $x$ -axis at  $x = 3$ , as shown in the figure above. On the interval  $0 < x < 6$ , the function  $f$  is twice differentiable, with  $f''(x) > 0$ .
  - a. Is  $f$  differentiable at  $x = 0$ ? Use the definition of the derivative with one-sided limits to justify your answer.
  - b. For how many values of  $a$ ,  $-4 \leq a < 6$ , is the average rate of change of  $f$  on the interval  $[a, 6]$  equal to 0?
  - c. Is there a value of  $a$ ,  $-4 \leq a < 6$ , for which the Mean Value Theorem, applied to the interval  $[a, 6]$ , guarantees a value  $c$ ,  $a < c < 6$ , at which  $f'(c) = \frac{1}{3}$ ? Justify your answer.
  - d. The function  $g$  is defined by  $g(x) = \int_0^x f(t) dt$  for  $-4 \leq x \leq 6$ . On what intervals contained in  $[-4, 6]$  is the graph of  $g$  concave up? Explain your reasoning.

2. ♦ The wind chill is the temperature, in degrees Fahrenheit ( $^{\circ}\text{F}$ ), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity  $v$ , in miles per hour (mph). If the air temperature is  $32^{\circ}\text{F}$ , then the wind chill is given by  $W(v) = 55.6 - 22.1v^{0.16}$  and is valid for  $5 \leq v \leq 60$ .
- Find  $W'(20)$ . Using correct units, explain the meaning of  $W'(20)$  in terms of the wind chill.
  - Find the average rate of change of  $W$  over the interval  $5 \leq v \leq 60$ . Find the value of  $v$  at which the instantaneous rate of change of  $W$  is equal to the average rate of change of  $W$  over the interval  $5 \leq v \leq 60$ .
  - Over the time interval  $0 \leq t \leq 4$  hours, the air temperature is a constant  $32^{\circ}\text{F}$ . At time  $t = 0$ , the wind velocity is  $v = 20$  mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at  $t = 3$  hours? Indicate units of measure.
3. Let  $f$  be the function given by  $f(x) = kx^2 - x^3$ , where  $k$  is a positive constant. Let  $R$  be the region in the first quadrant bounded by the graph of  $f$  and the  $x$ -axis.
- Find all values of the constant  $k$  for which the area of  $R$  equals 2.
  - For  $k > 0$ , write, but do not evaluate, an integral expression in terms of  $k$  for the volume of the solid generated when  $R$  is rotated about the  $x$ -axis.
  - For  $k > 0$ , write, but do not evaluate, an expression in terms of  $k$ , involving one or more integrals, that gives the perimeter of  $R$ .

4. Let  $R$  be the region in the first quadrant enclosed by the graphs of  $y = 2x$  and  $y = x^2$ , as shown in the figure above.
- Find the area of  $R$ .
  - The region  $R$  is the base of a solid. For this solid, at each  $x$  the cross section perpendicular to the  $x$ -axis has area  $A(x) = \sin\left(\frac{\pi}{2}x\right)$ . Find the volume of the solid.
  - Another solid has the same base  $R$ . For this solid, the cross sections perpendicular to the  $y$ -axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.



5. A solid is constructed so that it has a circular base of radius  $r$  centimeters and every plane section perpendicular to a certain diameter of the base is a square, with a side of the square being a chord of the circle.
- Find the volume of the solid.
  - If the solid expands so that the radius of the base increases at a constant rate of 0.5 centimeters per minute, how fast is the volume changing when the radius is 4 centimeters?
6. Let  $f$  be the function satisfying  $f'(x) = -3xf(x)$ , for all real numbers  $x$ , with  $f(1) = 4$  and  $\lim_{x \rightarrow \infty} f(x) = 0$ .
- Evaluate  $\int_1^{\infty} -3xf(x)dx$ . Show the work that leads to your answer.
  - Use Euler's Method, starting at  $x = 1$  with a step size of 0.5, to approximate  $f(2)$ .
  - Write an expression for  $y = f(x)$  by solving the differential equation  $\frac{dy}{dx} = -3xy$  with the initial condition  $f(1) = 4$ .
7. Consider the differential equation  $\frac{dy}{dx} = 1 - y$ . Let  $y = f(x)$  be the particular solution to this differential equation with the initial condition  $f(1) = 0$ . For this particular solution,  $f(x) < 1$  for all values of  $x$ .
- Use Euler's method, starting at  $x = 1$  with two steps of equal size, to approximate  $f(0)$ . Show the work that leads to your answer.
  - Find  $\lim_{x \rightarrow 1} \frac{f(x)}{x^3 - 1}$ . Show the work that leads to your answer.
  - Find the particular solution  $y = f(x)$  to the differential equation  $\frac{dy}{dx} = 1 - y$  with the initial condition  $f(1) = 0$ .



8. The figure above shows the path traveled by a roller coaster car over the time interval  $0 \leq t \leq 18$  seconds. The position of the car at time  $t$  seconds can be modeled parametrically by

$$x(t) = 10t + 4 \sin t$$

$$y(t) = (20 - t)(1 - \cos t)$$

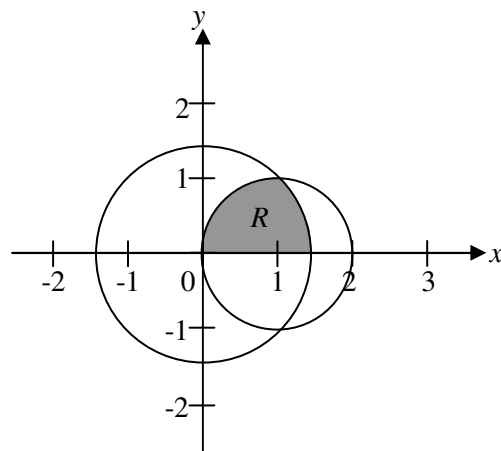
where  $x$  and  $y$  are measured in meters. The derivatives of these functions are given by

$$x'(t) = 10 + 4 \cos t$$

$$y'(t) = (20 - t) \sin t + \cos t - 1$$

- Find the slope of the path at time  $t = 2$ . Show the computations that lead to your answer.
- Find the acceleration vector of the car at the time when the car's horizontal position is  $x = 140$ .
- Find the time  $t$  at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.
- For  $0 < t < 18$ , there are two times at which the car is at ground level ( $y = 0$ ). Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

9. The figure at right shows the graphs of the circles  $x^2 + y^2 = 2$  and  $(x - 1)^2 + y^2 = 1$ . The graphs intersect at the points  $(1, 1)$  and  $(1, -1)$ . Let  $R$  be the shaded region in the first quadrant bounded by the two circles and the  $x$ -axis.



- Set up an expression involving one or more integrals with respect to  $x$  that represents the area of  $R$ .
- Set up an expression involving one or more integrals with respect to  $y$  that represents the area of  $R$ .
- The polar equations of the circles are  $r = \sqrt{2}$  and  $r = 2 \cos \theta$ , respectively. Set up an expression involving one or more integrals with respect to the polar angle  $\theta$  that represents the area of  $R$ .

10. Let  $S$  be the series  $S = \sum_{n=0}^{\infty} \left(\frac{t}{1+t}\right)^n$  where  $t \neq 0$ .

- Find the value to which  $S$  converges when  $t = 1$ .
- Determine the values of  $t$  for which  $S$  converges. Justify your answer.
- Find all values of  $t$  that make the sum of the series  $S$  greater than 10.

$$f(x) = \begin{cases} \frac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ \frac{-1}{2} & \text{for } x = 0 \end{cases}$$

11. The function  $f$ , defined above, has derivatives of all orders. Let  $g$  be the function defined by

$$g(x) = 1 + \int_0^x f(t) dt.$$

- Write the first three nonzero terms and the general term of the Taylor series for  $\cos x$  about  $x = 0$ . Use this series to write the first three nonzero terms and the general term of the Taylor series for  $f$  about  $x = 0$ .
- Use the Taylor series for  $f$  about  $x = 0$  found in part (a) to determine whether  $f$  has a relative maximum, relative minimum, or neither at  $x = 0$ . Give a reason for your answer.
- Write the fifth-degree Taylor polynomial for  $g$  about  $x = 0$ .
- The Taylor series for  $g$  about  $x = 0$ , evaluated at  $x = 1$ , is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for  $g$  about  $x = 0$  to estimate the value of  $g(1)$ . Explain why this estimate differs from the actual value of  $g(1)$  by less than  $\frac{1}{6!}$ .

## Answers

The answers below are *very* abbreviated. They are not intended to represent a complete response to the question.

- No
  - Two
  - $a = 3$
  - $(-4, 0) \cup (3, 6)$
- $-0.286$
  - $-0.254$ ;  $v = 23.011$
  - $-0.892$
- $k = \sqrt[4]{24}$
  - $\pi \int_0^k (kx^2 - x^3)^2 dx$
  - $k + \int_0^k \sqrt{1 + (2kx - 3x^2)^2} dx$
- $\frac{4}{3}$
  - $\frac{4}{\pi}$
  - $\int_0^4 \left( \sqrt{y} - \frac{y}{2} \right) dy$
- $\frac{16}{3} r^3$
  - 128
- $-4$
  - 2.5
  - $y = 4e^{3/2} e^{-3x^2/2}$
- $\frac{-5}{4}$
  - $\frac{1}{3}$
  - $f(x) = 1 - e^{1-x}$
- 1.794
  - $\langle -3.529, 1.226 \rangle$
  - $t = 3.024$ ; speed = 6.028
  - $t = 2\pi, 4\pi$
- $\frac{1}{2\pi} \int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos t)^2 + ((20 - t)\sin t + \cos t - 1)^2} dt$
  - $\int_0^1 \sqrt{1 - (x-1)^2} dx + \int_1^{\sqrt{2}} \sqrt{2 - x^2} dx$
  - $\int_0^{\pi/4} \frac{1}{2} (\sqrt{2})^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} (2\cos\theta)^2 d\theta$
- 2
  - $t > \frac{-1}{2}$
  - $t > 9$
- $f(x) = \frac{-1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \dots$
  - minimum
  - $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$
  - $g(1) \approx \frac{37}{52}$