PROBLEM SHEET 9 – Infinite Series

1. Given a differentiable function $f(x)$ that satisfies the differential equation $\frac{dy}{dx} = 2x + y$ and $f(1) = 4$.

   a. Find $\frac{d^2y}{dx^2}$ in terms of x and y.

   b. Write a second degree Taylor polynomial for $f(x)$ centered at $x = 1$.

   c. Use the second degree Taylor polynomial to approximate $f(1.5)$.

2. Given that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   a. Write out the first 5 terms of the expansion.

   b. Write out the first 5 terms and the general term of the expansion for e^{2x}.

   c. Write out the first 5 terms and the general term of the expansion for xe^x.

3. Given that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

   a. Write out the first 5 terms of the expansion.

   b. Write out the first 5 terms and the general term of the expansion for $\cos(x-5)$.

   c. Write out the first 5 terms and the general term of the expansion for $\frac{\cos x - 1}{x^2}$.

   d. Write out the first 5 terms and the general term of the expansion for $\sin x$. (Hint: Remember that the derivative of the sine function is the cosine function.)

4. Find the sum of the infinite series: $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

5. Find the sum of the infinite series: $1 + \frac{5}{1} + \frac{25}{2!} + \frac{125}{3!} + \frac{625}{4!} + \ldots$

6. Find the sum of the infinite series: $\sum_{n=0}^{\infty} \left[\left(\frac{2}{3} \right)^n \bullet \frac{1}{n!} \right]$.

7. Find the sum of the infinite series: $\sum_{n=0}^{\infty} \left[\frac{(-1)^n \bullet 2^{2n}}{5^{2n}(2n)!} \right]$.

8. Write out the first four terms of the Maclaurin series expansion of $\sin\left(3x - \frac{\pi}{4}\right)$.

Answers

1. a. $2 + 2x + y$
   b.
   c. 8

2. a. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

   b. $1 + 2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \frac{2^4 x^4}{4!} + \ldots \frac{2^n x^n}{n!} + \ldots$

   c. $x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \ldots \frac{x^{n+1}}{n!} + \ldots$

3.

4. $\frac{3}{4}$

5.

6. $e^{2/3}$

7.

8. first term is $\frac{-\sqrt{2}}{2}$; second term is $\frac{3\sqrt{2}}{2}x$