

PROBLEM SHEET 6 – Differential Equations

Some problems on this sheet are taken from *Calculus; Single Variable*, second edition by Hughes-Hallett, Gleason, et al. New York, John Wiley & Sons, 1998.

1. For each of the following functions, sketch by hand the direction field on a graph of size $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. Use your TI-CAS to check your result. Then try to guess a formula for the general solution. (Remember to load the *Plot Slope field file*.)

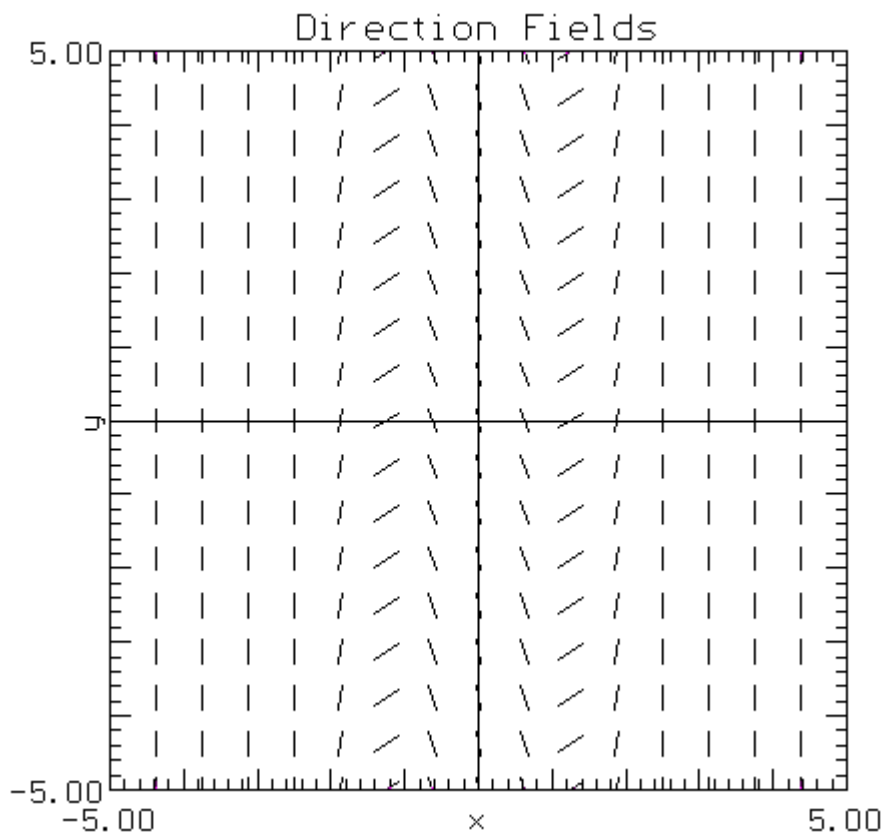
a. $\frac{dy}{dx} = x - 1$

b. $\frac{dy}{dx} = y$

c. $\frac{dy}{dx} = \frac{-x}{y}$

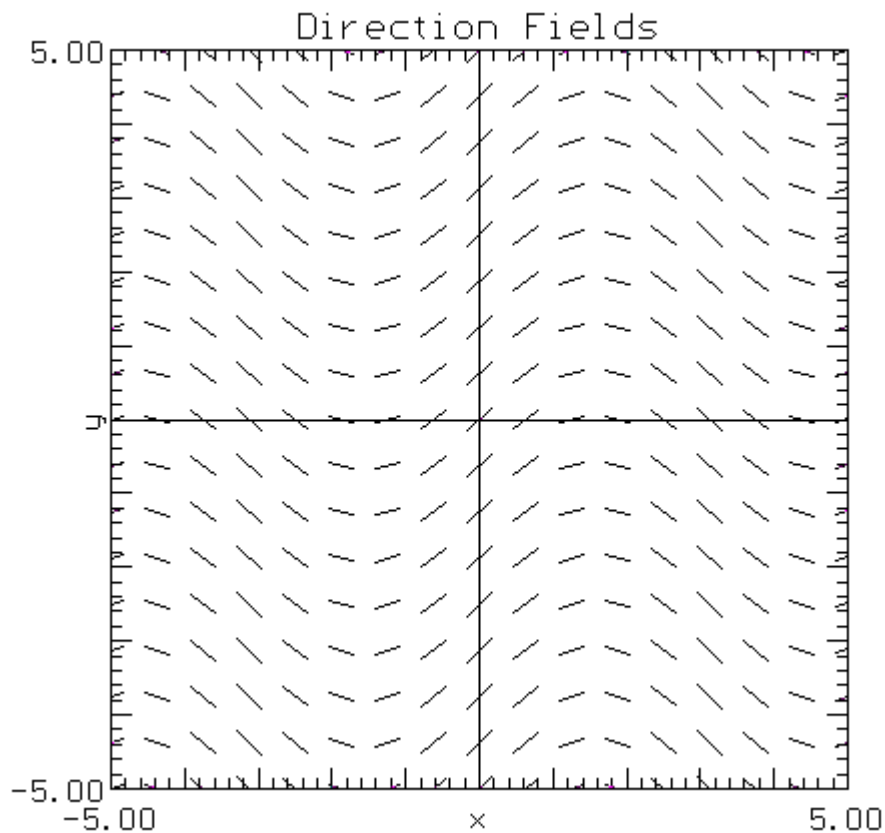
2. For the direction field at right:

- a. Draw a specific solution where $f(-2) = 0$.
- b. Try to guess a formula for the general solution.



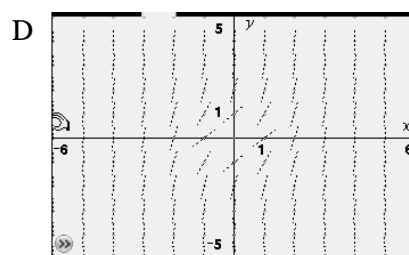
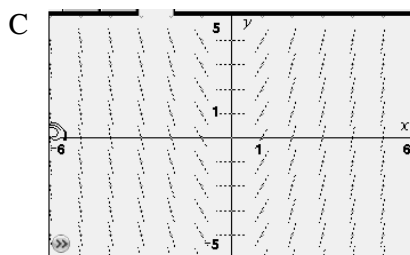
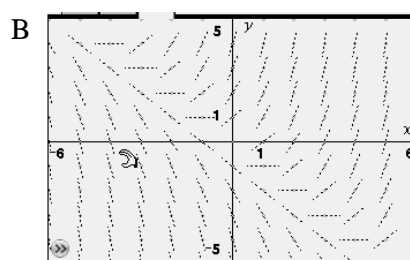
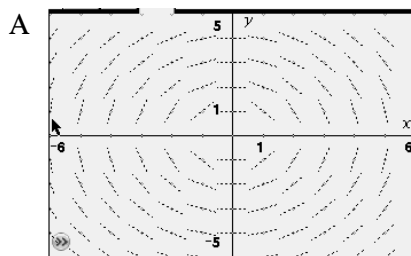
3. For the direction field at right:

- a. Draw a specific solution where $f(-\pi) = 0$.
- b. Try to guess a formula for the general solution.



Match the differential equation with its slope field.

- 4. $y' = 2x$
- 5. $y' = x + y$
- 6. $y' = x^2 + y^2$

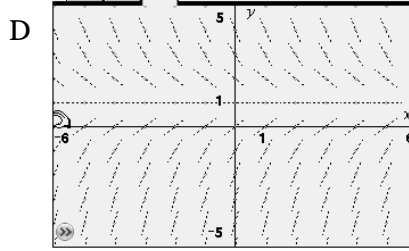
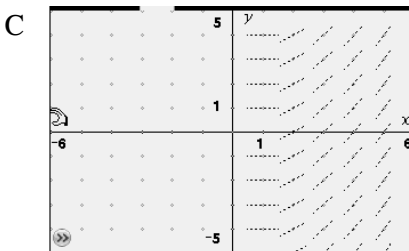
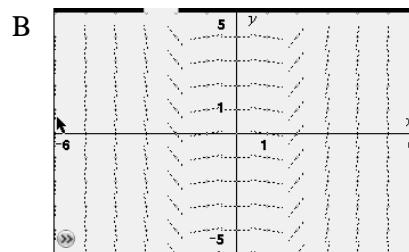
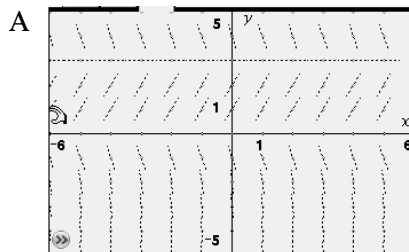


Match the differential equation with its slope field.

7. $y' = 1 - y$.

8. $y' = y(3 - y)$

9. $y' = \ln x$



10. Use Euler's Method to generate points for the function defined by the differential equation $\frac{dy}{dx} = 2x$ with initial condition that $y = 0$ when $x = 0$. Calculate 8 values for y using $\Delta x = 0.5$. Do this problem completely BY HAND - no TI-CAS at all.

Recall that the essence of Euler's Formula is that:
 (new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).
 We know that the starting point is $(0, 0)$. So $x_0 = 0$ and $y_0 = 0$.

Then $y_1 = y_0 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0 + (2 * 0) \bullet (0.5) = 0$
 $y_2 = y_1 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0 + (2 * 0.5) \bullet (0.5) = 0.5$
 $y_3 = y_2 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0.5 + (2 * 1) \bullet (0.5) = 1.5$

Continue from there to generate the remaining points asked for.

11. Given the differential equation $\frac{dP}{dt} = 0.2P$ with initial conditions $P = 1$ when $t = 0$, use Euler's Method to generate values to approximate the function from $t = 0$ to $t = 3$ using $\Delta t = 0.3$. Record your values in a table.

Recall, Euler's Method says that:
 (new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).
 Expressing this same idea in calculus notation we get: $y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$.
 Substituting in the information from this problem yields:
 $y_{i+1} = y_i + (0.2 \bullet y_i) \bullet 0.3$ with starting point $(0, 1)$ since $P = 1$ when $t = 0$.

12. The change in the velocity of a body falling at a relatively slow speed over a short distance is given by $\frac{dv}{dt} = g - kv$, where g is the acceleration due to gravity and k is a constant. Let $g = 9.8$ m/sec², $k = 0.02$, $\Delta t = 2$, and $v_0 = 0$.
- Write a formula for velocity using Euler's Method.
 - Calculate points for the velocity function over the next 50 seconds. Use the same procedure as in the previous problem. Enter a starting value of 0 in your calculator. Then enter Euler's formula as it relates to this problem, using *ANS* everywhere the v_i variable appears. Now with repeated pressing of the *ENTER* key you should get the appropriate values. Record the values for a time of 20 seconds and a time of 50 seconds.
13. Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding air (or water). Let T be the constant temperature of the surroundings, y be the temperature of the object, and t be the time. Then symbolically the above statement says: $\frac{dy}{|y-T|} = k$ or $\frac{dy}{dt} = k|y-T|$.
- Use this expression and Euler's Method to write an expression for y_{i+1} .
 - Suppose a hot object is plunged into a beaker of ice water that is maintained at a constant temperature of 32°F. Let $k = -0.1$, $y_0 = 250^\circ$, and $\Delta t = 0.5$ minutes. Use Euler's Method and your TI-CAS to determine the temperature of the object 30 minutes later.

For problems # 14 – 17, find the solutions to the differential equation, subject to the given initial condition, where given.

14. $\frac{dy}{dx} + \frac{y}{3} = 0$, and $y(0) = 10$.

15. $\frac{dz}{dr} = z + zr^2$, and $z(0) = 1$.

16. $\frac{dP}{dt} = P - a$. Assume that a is constant.

17. When the electromotive force (emf) is removed from a circuit containing inductance and resistance but no capacitors, the rate of decrease of current is proportional to the current. If the initial current is 30 amps but decays to 11 amps after 0.01 seconds, find an expression for the current as a function of time.
18. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For example, this is the case as δ -glucono-lactone changes into gluconic acid.
- Write a differential equation satisfied by y , the quantity of δ -glucono-lactone present at time t .
 - If 100 grams of δ -glucono-lactone is reduced to 54.9 grams in one hour, how many grams will remain after 10 hours?

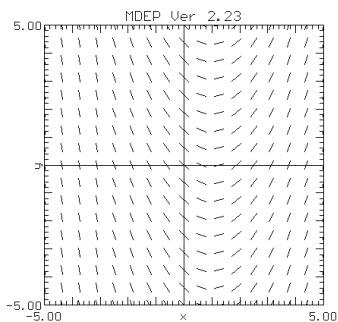
19. The rate (per foot) at which light is absorbed as it passes through water is proportional to the intensity, or brightness, at that point.
- Find the intensity as a function of the distance the light has traveled through the water.
 - If 50% of the light is absorbed in 10 feet, how much is absorbed in 20 feet? 25 feet?
20. The radioactive isotope carbon-14 is present in small quantities in all life forms. It is constantly replenished until the organism dies, after which it decays to stable carbon-12 at a rate proportional to the amount of carbon-14 present, with a half-life of 5730 years. Suppose $C(t)$ is the amount of carbon-14 present at time t .
- Find the value of the constant k in the differential equation $C' = kC$.
 - In 1988 three teams of scientists found that the Shroud of Turin, which was reputed to be the burial cloth of Jesus, contained 91% of the amount of carbon-14 contained in freshly made cloth of the same material. How old is the Shroud of Turin, according to these data?
21. Warfarin is a drug used as an anticoagulant. After stopping administration of the drug, the quantity remaining in a patient's body decreases at a rate proportional to the quantity remaining. The half-life of Warfarin in the body is 37 hours.
- Sketch a rough graph of the quantity, Q , of Warfarin in a patient's body as a function of the time, t , since stopping administration of the drug. Mark the 37 hours on your graph.
 - Write a differential equation satisfied by Q .
 - Solve the differential equation.
 - How many days does it take for the drug level in the body to be reduced to 25% of the original level?
22. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3° F. One hour later, the temperature of the body is 89.0° F. The temperature of the room has been maintained at a constant 68° F.
- Assuming the temperature, y , of the body obeys Newton's Law of Cooling, write a differential equation for y .
 - Solve the differential equation to estimate the time the murder occurred.
23. What are the equilibrium solutions for the differential equation $\frac{dy}{dt} = 0.2(y - 3)(y + 2)$?
24. Find the equilibrium solution to the differential equation $\frac{dy}{dt} = 0.5y - 250$.

25. The growth of a certain animal population is governed by the equation $\frac{dP}{dt} = \frac{P}{1000}(100 - P)$, where $P(t)$ is the number of animals in the colony at time t . The initial population is known to be 200 animals.
- Use Euler's Method to write a recursive formula for P .
 - Use Euler's Method with $\Delta t = 0.5$ days to determine approximately when will there be 150 animals?
 - What are the maximum and minimum values the population will attain?
 - Solve the differential equation analytically.
 - When will the number of animals be within 2 of the minimum?
26. Prove that the fastest rate of growth on a logistics curve with $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$ occurs when $y = \frac{L}{2}$.
(Hint: Rewrite the derivative as $\frac{dy}{dt} = ky - \frac{k}{L}y^2$. Differentiate implicitly to get an expression for the second derivative. Set this second derivative equal to zero and solve for y .)
27. Lake Michigan contains 4.9 thousand cubic kilometers of water and has an outflow of 158 cubic kilometers per year. How long would it take for 90% of the pollution to be removed from the lake?
28. Money in a bank account grows continuously at an annual rate of r (when the interest rate is 5%, $r = 0.05$, and so on). Suppose \$1000 is put into the account in 1970.
- Write a differential equation satisfied by M , the amount of money in the account at time t , measured in years since 1970.
 - Solve the equation.
 - Sketch a graph of the solution until the year 2000 for interest rates of 5% and 10%.
29. A certain national park is known to be able to support at most 100 grizzly bears. Ten bears are known to be in the park in 1980. In 1990 there were 25 bears in the park. Assume that the bear population growth is logistic in nature.
- Write a differential equation to describe the growth of the bear population.
 - Solve the differential equation.
 - When will there be 75 bears in the park?
 - When will the bear population be growing most rapidly?
30. As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student forgets material is proportional to the difference between the material he or she currently remembers and some positive constant, a .

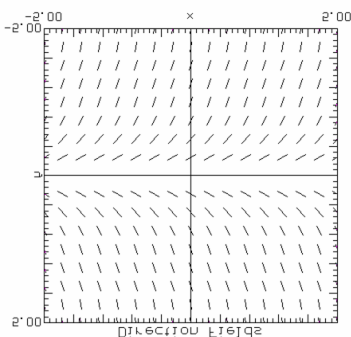
- a. Let $y = f(t)$ be the fraction of the original material remembered t weeks after the course has ended. Set up a differential equation for y . Your equation will contain two constants. The constant a is less than y for all t .
- b. Solve the differential equation.
- c. Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$.
31. A small town charity fund drive aims to raise \$65,000. Updated current totals are posted in the town square. According to Alfred E. Neuman's law of cooling enthusiasm, the rate at which people contribute to such a drive is proportional to the difference between the current total and the announced target amount. Let $y(t)$ represent the current total, in thousands of dollars, t weeks after the start of the drive. Suppose that after 6 months they have collected \$40,000.
- a. Does Neuman's law of cooling enthusiasm sound reasonable? Why or why not?
- b. Express Neuman's law of cooling enthusiasm as a differential equation.
- c. Solve the differential equation.
- d. How long will it take for this town to be within \$5,000 of their goal?
32. TI-CAS. For the years 1985 through 1994, the rate of consumption of beef (in billions of pounds) in the United States can be modeled by $f(t) = \begin{cases} 27.77 - 0.36t & 5 \leq t \leq 10 \\ 21.00 + 0.27t & 10 \leq t \leq 14 \end{cases}$ where t is the time in years, with $t = 5$ corresponding to 1985. Suppose the rate of beef consumption for 1990 through 1994 had continued to follow the model for the years 1985 through 1990. How much less beef would have been consumed from 1990 through 1994?
33. TI-CAS. For the years 2000 to 2010, the projected rate of fuel cost C (in millions of dollars) for a corporation is $C_1 = 568.50 + 7.15t$ where t is the time in years, with $t = 0$ corresponding to 2000. Because of the installation of fuel-saving equipment, a more accurate model of the rate of fuel costs for the period is $C_2 = 525.60 + 6.43t$. Approximate the savings for the 10-year period owing to the installation of the new equipment.

Answers

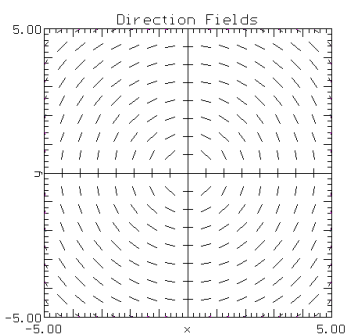
1. a. $y = \frac{1}{2}x^2 - x + c$



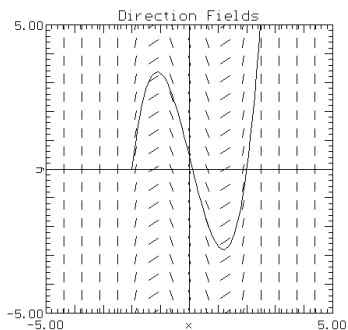
b. $y = e^x + c$



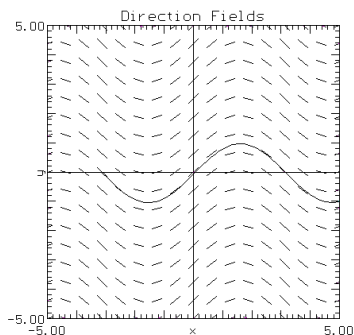
c. $x^2 + y^2 = r^2$



2. $y = x^3 - 4x$



3. $y = \sin(x)$



4. C

5. B

6. D

7.

8.

9.

10.

x	y
0	0
0.5	0
1	0.5
1.5	1.5
2	3
2.5	5
3	7.5
3.5	10.5

11. $P_{i+1} = P_i + \frac{dP}{dt}(\Delta t)$

t	P
0	1
0.3	1.06
0.6	1.1236
0.9	1.1910
1.2	1.2625
.	
.	
.	

12. a.
13. a.
b. 42.043°F

t	y
0	250
0.5	239.1
1	228.75
1.5	218.91
2	209.56
.	.

14. $y = 10e^{-x/3}$

15. $z(r) = e^{r+r^3/3}$

16.

17. $C(t) = 30e^{-100.33t}$

18. a. $\frac{dy}{dt} = ky$

b. $y = 100e^{t \ln 0.549}$; $y = 0.249$

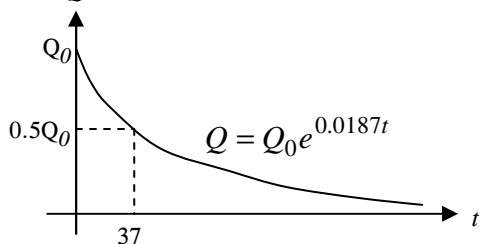
19. a. $\frac{dI}{ds} = kI$; $I = I_0 e^{ks}$ Where I is the intensity of light remaining.

b. $I = I_0 e^{((1/10) \ln 0.5)s}$; At $s = 20$, $\frac{I}{I_0} = 0.25$. Hence 75% has been absorbed.

20. a. $k = (1/5730) \ln 0.5$

b. $0.91 = e^{(t/5730) \ln 0.5}$; 779.6 years

21. a.



b.

c. $Q = Ce^{((1/37) \ln 0.5)t}$

d. 3 days

22. a. $\frac{dy}{dt} = k(y - T)$

b. $y = 68 + 22.3e^{t \cdot \ln(21/22.3)}$ where $t = 0$ at the time the body is found.

The murder occurred at 3:44 AM.

23. $\{-2, 3\}$

24. $\{500\}$

25. a. $P_{i+1} = P_i + \frac{P_i}{1000}(100 - P_i)\Delta t$

b. about 4 days

c. max is 200, min is 100

d. $P = \frac{100}{1 - 0.5e^{-t/10}}$

e. about 31 days

26.

27. 71 years

28. a. $\frac{dM}{dt} = rM$

b. $M = 1000e^{rt}$

29. a. $\frac{dB}{dt} = kB \left(1 - \frac{B}{100}\right)$

b. $B = \frac{100}{1 + 9e^{-0.11t}}$

c. about 30 years, or the year 2100

30. a.

b. $y = (1-a)e^{kt} + a$

c. a is the amount remembered over a long period of time.

k is the rate at which the material is forgotten.

31.

32. c. 3.16 billion pounds

33. 465 million dollars