Major Themes of Calculus Sure to Appear on the Semester Examination

Limits

- A limit is the single value to which an expression approaches as the x value approaches infinity or a real number.
- A limit can be evaluated by substitution if it yields a determinate form.
- If the limit is indeterminate upon substitution, the expression needs to be simplified using factoring or multiplying by the conjugate. If the expression contains an absolute value, it is often best to examine the left-hand and right-hand limits separately.
- Both left and right-hand limits must exist and be equal for the limit to exist at a point.

Continuity

- In general, continuity refers to the graph being "connected".
- To be continuous at a point: 1. the function must be *defined* at the point, 2. the *limit must exist* at the point, 3. the *value of the limit must be the defined value* of the function at that point.
- When checking to see if a piecewise function is continuous: 1. check that each separate piece is continuous over its restricted domain, 2. check that the function is defined at the connecting point between the two branches, 3. check that the limit of the left branch of the curve and the limit of the right branch of the curve are both equal to the value of the function at the connecting point.

Difference Quotients

- A difference quotient on a function y is the change in y over the change in x.
- A difference quotient gives the slope between *two* fixed points on a graph.
- The difference quotient on a function y gives the *average* rate of change of y with respect to x.
- If y represents a position or distance and x represents time, then the difference quotient gives the *average* velocity.

Derivatives

- The derivative is the *limit* of a different quotient as the change in x approaches zero.
- The derivative of a function gives the slope of a curve at a *single* fixed point.
- The derivative of y gives the *instantaneous* rate of change of y with respect to x.

- If y represents a position or distance, then the derivative of y gives the *instantaneous* velocity.
- If y represents a velocity function, then the derivative of y gives the *instantaneous* acceleration.
- A derivative can not exist where a function is undefined, discontinuous, has a corner on its graph, or has a vertical tangent line.
- To determine if a piecewise defined function is differentiable: 1. check that each branch is differentiable over its restricted domain, 2. check that the function is *continuous* at the connecting point, 3. check that the *limit of the derivative* of the left branch equals the *limit of the derivative* of the right branch at the connecting point.
- All differentiable curves are continuous.
- If the derivatives of two different functions are equal, then the functions differ only by a constant.

Tangent Lines

- The slope of the tangent line at a point is the value of the derivative at that point.
- A tangent line is used as a linear approximation of a curve near the point of tangency.
- The differential uses the tangent line to approximate the change in y for a given change in x.

Relating Derivatives to Graphs

- If the derivative of a function is positive, the function is increasing; if the derivative of a function is negative, the function is decreasing.
- To find critical points, find all points (in the domain of the function) where the derivative is zero or undefined.
- If a function has a critical point where the derivative changes from positive to negative, the point is a relative maximum. If a function has a critical point where the derivative is zero and the function is concave downward, the point is a relative maximum.
- If a function has a critical point where the derivative changes from negative to positive, the point is a relative minimum. If a function has a critical point where the derivative is zero and the function is concave upward, the point is a relative minimum.
- To find an *absolute* maximum or minimum, find all relative extrema and compare their y-values with any endpoints on the function. If the function does not have endpoints, do one of the following: 1. compare the relative extrema with the limit of the function as x approaches positive infinity and as x approaches negative infinity, or 2. make an argument based on where the function is increasing or decreasing.

- To find points of inflection, find all points (in the domain of the function) where the second derivative is zero or undefined *and* where the second derivative changes sign on either side of the point.

- If the second derivative is positive, the graph is concave upward. If the second derivative is negative, the graph is concave downward. Visually, if the tangent line is below the graph, the graph is concave upward. If the tangent line is above the graph, the graph is concave downward.

Definite Integrals

- The definite integral is the limit of a summation of terms, called a Riemann Sum. Each term is the product of the y value of the function times a small interval of change in x. The limit is taken as the interval of change approaches zero.

- The First Fundamental Theorem of Calculus tells us that the definite integral can be evaluated by an entirely different method, using the antiderivative of the function.

- The Second Fundamental Theorem of Calculus tells us that if we let the upper bound on the definite integral of f be a variable, the new function generated will be an antiderivative of f. We call this antiderivative an accumulation function because it accumulates positive and negative area under the curve as x moves along the axis.

- The definite integral gives the area under a curve, provided the curve is always above the x-axis.

- The definite integral of a *rate of change function, meaning* f', gives the *total change* in the original function f. If you need the *total amount*, you need to add this change to the starting amount.

- All continuous curves are integrable.

Indefinite Integrals

- The indefinite integral represents the family of all antiderivatives of a function. Remember that the family of antiderivatives is expressed with a $+C$ on the end.

- Techniques for finding an indefinite integral:
  a. observe a power function structure (with the appropriate factor necessary for the chain rule).
  b. observe an inverse trig function structure.
  c. use a substitution (especially for a radical expression).
  d. divide numerator into denominator (when the power of the numerator is the same or higher than the power of the denominator).
  e. split the function into separate terms.
  f. use the process of partial fractions to separate a rational function into individual fractions.
  g. use a Math Analysis trig formula to replace a difficult trigonometric expression.

Differential Equations

- A differential equation is any equation that involves both functional expressions and derivative expressions. The solution of a differential equation is a function *defined on an open interval* which makes the differential equation true.

- To solve a differential equation analytically, separate the variables so that each is on one side of the equation. Then integrate both sides of the equation, remembering to write $+C$. If a particular point on the function is given, use it to find the value of C. Remember that when you are finding a particular solution, your answer must be a *function*, not a relation.

- To solve a differential equation numerically, write a recursive formula to express the fact that the new y value equals the previous y value plus the change in y. The change in y is approximated by the differential of y. Hence $y_{i+1} = y_i + y'_i \bullet \Delta y$ which is Euler's formula.

- A logistic differential equation is identified by the presence of two linear factors both involving the dependent variable and a coefficient, such as $ky\left(1-\frac{y}{L}\right)$ or $ky(L-y)$. The graph of a logistic function is monotonic and has two horizontal asymptotes. The point of inflection is always midway between the two asymptotes

Applications of Integration

- The definite integral of a velocity function gives the total *change in position* of the object. It DOES NOT give the total *distance* traveled unless the velocity has all been in one direction (that is, all positive or all negative) If the velocity changes direction, you get the total distance by integrating the absolute value of the function, or by splitting the integral up into pieces over which it does not change sign.

- The area under a curve is found by integrating the function, provided the function is ABOVE the x-axis. If the function is sometimes below the y-axis, integrate the *absolute value* of the function, or split it into separate integrals adjusting the sign as needed.

- Area between two curves is found by integrating the difference of the two curves; that is subtracting the lower curve from the upper one.

- The average value of a function is found by taking the integral of the function and dividing it by the width of the interval. The average value corresponds to the height of a rectangle that has the same area as the area under the curve.

- To find the volume of a solid by slicing, find the volume of one individual slice. Then integrate this expression (which has the effect of adding the volumes of all slices and taking the limit as the thickness of each slice approaches 0.)

- To find the volume of a solid of revolution, the figure is sliced *perpendicular* to the axis of rotation. Each slice is a cylinder; hence the volume of each slice can be expressed as $\pi r^2 h$, where h is the thickness of the slice. Then integrate this expression.