DEFINITIONS, THEOREMS, AND FORMULAS – Semester 2

Chapter 8 – Integration Techniques

❖ ***Indicates that the item should be memorized in exact detail. You may be asked to quote it on a quiz or test.***

1. Thm. ❖**Integration by Parts**: $\int u\,dv = uv - \int v\,du$.

2. Thm. ❖**L'Hôpital's Rule**: If $\lim\left[\dfrac{f(x)}{g(x)}\right]$ results in the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\left[\dfrac{f(x)}{g(x)}\right] = \lim\left[\dfrac{f'(x)}{g'(x)}\right]$ provided the latter limit is of a determinate form.

3. Def. **Improper Integral with Infinite Integration Limit**:

   a. If f is continuous on the interval $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

   b. If f is continuous on the interval $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$

   c. If f is continuous on the interval $(-\infty, \infty)$, then $\displaystyle\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$,

   where c is any real number.

   Note that for the first two cases, the improper integral **converges** if the limit exists; otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if *either* of the improper integrals on the right diverges.

4. Def. **The Factorial Function**: $\displaystyle x! = \int_0^\infty t^x e^{-t}\,dt$.

5. Def. **Improper Integral with Infinite Discontinuity (or Vertical Asymptote)**:

   a. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity (or vertical asymptote) at b, then $\displaystyle\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$.

   b. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity (or vertical asymptote) at a, then $\displaystyle\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$.

c. If f is continuous on the interval $[a, b]$ except for some point $c \in (a, b)$ at which f has an infinite discontinuity (or vertical asymptote), then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

Note that for the first two cases, the improper integral converges if the limit exists; otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges if *either* of the improper integrals on the right diverges.

6. Def. **Piecewise Continuity**: Function f is piecewise-continuous on the interval $[a, b]$ if and only if there is a finite number of values of x in $[a, b]$ at which $f(x)$ is discontinuous, the discontinuities are either removable or step discontinuities, and f is continuous elsewhere on $[a, b]$.

7. Thm. **Integrability of a Piecewise Continuous Function**: If function f is piecewise-continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.

--*End for Chapter 8 Test*--

Chapter 10 – Parametric Functions

8. Def. **Parametric Equations**: If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and t is called the parameter.

9. Def. **Orientation**: When a parametric curve is plotted in order of increasing values of the parameter, the curve is traced out in a specific direction, called the orientation of the curve. Orientation is also referred to as the *direction of motion*.

10. Def. **Intersection Point:** Given two parametrized curves, an intersection point is any point that lies on both curves. If this intersection point corresponds to the same value of t for both curves, it is called a *collision point*.

11. Def. **Vector**: A vector is a quantity which is described by a direction and a magnitude. It is represented by a ray $\overrightarrow{AB}$, where A is called the *initial point* (or tail of the vector) and B is called the *terminal* point (or head of the vector).

12. Def. **Magnitude (or Absolute Value) of a Vector**: $\left\| \vec{v} \right\|$ is called *the* magnitude or *absolute* value of the vector. Its value corresponds to the length of the arrow, with no regard for direction.

13. Def. **Zero Vector**: The zero vector is a vector of magnitude zero, i.e. a point. (It is assigned no direction.)

14. Def. **Scalar Multiple of a Vector**: $k\vec{v}$ (where $k > 0$) represents a vector with the same direction as $\vec{v}$, but whose magnitude is k times as large.

15. Def. **Component Form of a Vector**: Let the tail of a vector be at (x_1, y_1) and its head be at (x_2, y_2). Then the vector $\vec{v}$ can be expressed as $\vec{v} = \langle a, b \rangle$ where $a = x_2 - x_1$ and $b = y_2 - y_1$.

16. Def. **Opposite of a Vector**: $-\left(\vec{AB}\right)$ means $\vec{BA}$.

17. Thm. **Vector Operations in Components and Unit Vectors:** For $\vec{v} = \langle a, b \rangle = a\,\vec{i} + b\,\vec{j}$, $\vec{w} = \langle c, d \rangle = c\,\vec{i} + b\,\vec{j}$ and scalar k:

    a. $\left\|\vec{v}\right\| = \sqrt{a^2 + b^2}$

    b. $k\,\vec{v} = \langle ka, kb \rangle = ka\,\vec{i} + kb\,\vec{j}$

    c. $\vec{v} + \vec{w} = \langle a + c, b + d \rangle = (a + c)\,\vec{i} + (b + d)\,\vec{j}$

    d. $\vec{v} - \vec{w} = \langle a - c, b - d \rangle = (a - c)\,\vec{i} + (b - d)\,\vec{j}$

18. Thm. **Parallel Vectors**: $\vec{v}$ and $\vec{w}$ are parallel if and only if $\vec{w} = k\,\vec{v}$.

19. Def. **Dot Product**: The dot product of two vectors $\vec{v} = \langle a, b \rangle$ and $\vec{w} = \langle c, d \rangle$ is $\vec{v} \bullet \vec{w} = ac + bd$. (Note that the dot product of two vectors is always a *scalar*.)

20. Thm. **Orthogonal Vectors**: $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \bullet \vec{w} = 0$.

21. Thm. **Displacement Formula**: Displacement equals velocity times time: $\vec{d} = \vec{v}\,t$.

22. Thm. **Position Formula**: $(x,y) = (x_0,y_0) + \vec{v}\,t$ describes the position of object moving with given velocity $\vec{v}$ from a given starting point (x_0, y_0). Note: The Position Formula $(x, y) = (x_0, y_0) + t\langle a, b \rangle$ can be expressed as a pair of parametric equations: $x = x_0 + ta$ and $y = y_0 + tb$, where t is the parameter.

23. Def. **Velocity Vector**: If the position of an object is determined by two parametric functions $x(t)$ and $y(t)$, or a vector function $\langle x(t), y(t) \rangle$, the vector $\langle x'(t), y'(t) \rangle$ is called the velocity vector of the object at any time t. Individually, $x'(t)$ is called the *instantaneous velocity in the x-direction*; and $y'(t)$ is called the *instantaneous velocity in the y-direction*.

24. Def. **Instantaneous Speed**: If the position of an object is determined by two parametric functions $x(t)$ and $y(t)$, or a vector function $\langle x(t), y(t) \rangle$, the instantaneous speed of the object is defined to be the magnitude of the velocity vector. That is, $speed = \sqrt{(x'(t))^2 + (y'(t))^2}$.

25. Def. **Acceleration Vector**: If the position of an object is determined by two parametric functions $x(t)$ and $y(t)$, or a vector function $\langle x(t), y(t)\rangle$, the vector $\langle x''(t), y''(t)\rangle$ is called the acceleration vector of the object at any time t.

26. Thm. **Parametric Motion**: Given two parametric functions $x(t)$ and $y(t)$, or a vector function $\langle x(t), y(t)\rangle$, that describe the motion of an object. At any point t for which:

    a. $y'(t)=0$ and $x'(t)\neq 0$, the motion will be parallel to the x-axis and the curve will have a horizontal tangent.

    b. $y'(t)\neq 0$ and $x'(t)=0$, the motion will be parallel to the y-axis and the curve will have a vertical tangent.

    c. $y'(t)=0$ and $x'(t)=0$, the motion is stopped. If neither $x'(t)$ nor $y'(t)$ change sign at the point, the curve will have a tangent line. Otherwise there will be no tangent line.

27. Def. **Smooth Curve in Parametric Form**: A curve represented by parametric functions $x(t)$ and $y(t)$ on an interval I is called smooth if $x'(t)$ and $y'(t)$ are *continuous* on I and are *not simultaneously zero*, except possibly at the endpoints of I.

28. Thm. ❖**Parametric Form of the Derivative**: If a smooth curve is given by the parametric functions $x(t)$ and $y(t)$, then the slope of the curve at (x, y) is $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}$, provided that $\dfrac{dx}{dt}\neq 0$.

29. Thm. ❖**Second Derivative**: If a smooth curve is given by the parametric functions $x(t)$ and $y(t)$, then $\dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$, provided that $\dfrac{dx}{dt}\neq 0$.

30. Def. ❖**Length of a Parametrically Defined Path**: If an object travels along a smooth curve given by the parametric functions $x(t)$ and $y(t)$, or a vector-valued function $\langle x(t), y(t)\rangle$ then the length of the path traveled by the object during the time interval $a\le t\le b$ is given by $s=\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$.

31. Thm. **Arc Length of a Parametrically Defined Curve**: If a curve that is defined by the parametric functions $x(t)$ and $y(t)$, or a vector-valued function $\langle x(t), y(t)\rangle$, does *not intersect itself* on the interval $a\le t\le b$ (except possibly at the endpoints), then the arc length of the curve from $(x(a), y(a))$ to $(x(b), y(b))$ is given by

    $$s=\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt .$$

32. Def. **Vector-Valued Functions**: A function of the form $\vec{r}(t)=f(t)\,\vec{i}+g(t)\,\vec{j}$ is called a vector-valued function, where the component functions f and g are real-valued functions of the parameter t. Unless otherwise stated, the domain of a vector-valued function $\vec{r}$ is

considered to be the intersection of the domains of the component functions f and g. (Note that vector-valued functions are also expressed in the form $\vec{r}(t) = \langle f(t), g(t) \rangle$ or $\vec{r}(t) = (f(t), g(t))$.)

33. Def. **Limit of a Vector-Valued Function**: If $\vec{r}$ is a vector-valued function such that $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$, then $\lim\limits_{t \to a} \vec{r}(t) = \left[\lim\limits_{t \to a} f(t)\right]\vec{i} + \left[\lim\limits_{t \to a} g(t)\right]\vec{j}$, provided that f and g both have limits as t approaches a.

34. Thm. **Continuity of a Vector-Valued Function**: A vector-valued function $\vec{r}$ is continuous at the point determined by $t = a$ if and only if each of its component functions is continuous at the point determined by $t = a$.

35. Thm. **Differentiation of a Vector-Valued Function**: If a vector-valued function is defined by $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$, where f and g are differentiable functions of *t*, then $\vec{r}'(t) = f'(t)\,\vec{i} + g'(t)\,\vec{j}$.

36. Thm. **Tangent Vector**: Let $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ be a vector-valued function where $x(t)$ and $y(t)$ represent the rectangular coordinates of a curve C. Then $\vec{r}'(t)$ represents a vector that will be tangent to the curve C at the point $(x(t), y(t))$. *Application*: For a particle moving along a curve, the velocity vector of the particle's motion will be tangent to the path of the particle at each point.

--*End for Chapter 10 Quiz*--

37. Thm. **Converting Polar to Cartesian (Rectangular)**: The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows:

a. $y = r\sin\theta$ b. $x = r\cos\theta$

c. $r = \sqrt{x^2 + y^2}$ d. $\tan\theta = \dfrac{y}{x}$

38. Thm. **Classification of Conics**: If P is any point on the curve, F is a focus and D is the related directrix, then a conic is a(n)

a. Ellipse if $0 < \dfrac{|PF|}{|PD|} < 1$ b. Parabola if $\dfrac{|PF|}{|PD|} = 1$

c. Hyperbola if $\dfrac{|PF|}{|PD|} > 1$

39. Def. **Eccentricity**: If P is any point on the curve of a conic with center C, one focus F, and its directrix D, then the eccentricity e of a conic section is given by $e = \dfrac{|\mathrm{CF}|}{|\mathrm{CV}|} = \dfrac{|\mathrm{PF}|}{|\mathrm{PD}|}$.

40. Thm. **Polar Equations of Conics**: The graph of a polar equation of the form $r = \dfrac{ed}{1 \pm e\cos\theta}$ or $r = \dfrac{ed}{1 \pm e\sin\theta}$ is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus (which is at the pole) and its corresponding directrix.

41. Thm. **Slope in Polar Form**: If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r,θ) is $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$, provided that $\dfrac{dx}{d\theta} \neq 0$ at (r,θ).

(This derivative formula can be derived from the polar equations $x = r\cos\theta$ and $y = r\sin\theta$ using the parametric derivative formula $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$. Remember to use the Product Rule and Chain Rule when differentiating the functions for x and y because r is a function of θ.)

42. Thm. **Vertical and Horizontal Tangent Lines to a Polar Curve**: Given a polar curve $r = f(\theta)$. If at a point on the curve (r,θ):

a. $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$, then the tangent line to the curve at the point is horizontal.

b. $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$, then the tangent line to the curve at the point is vertical.

c. $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} = 0$, then no conclusion can be drawn.

43. Thm. **Tangents at the Pole**: Given a polar curve $r = f(\theta)$ that passes through the pole at point $(0,\alpha)$. The tangent line to the curve at $(0,\alpha)$ is $y = (\tan\alpha)x$ or $\theta = \alpha$, provided that $r'(\alpha) \neq 0$.

44. Thm. **Sector of a Circle**: The area of a sector of a circle of radius r with central angle θ (in radians) is defined by $A = \dfrac{\theta}{2\pi}\pi r^2 = \dfrac{1}{2}\theta r^2$.

45. Thm. ❖**Area in Polar Coordinates**: If $r = f(\theta)$ is continuous and non-negative on the interval $[\alpha,\beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta [f(\theta)]^2\,d\theta$

---*End for Chapter 10 Test*---

Chapter 9 – Infinite Series

46. Def. **Sequence**: A sequence is a function whose domain is the set of natural numbers.

47. Def. **Limit of an Infinite Sequence**: The sequence $\{a_n\}$ has a limit L, written $\lim_{n\to\infty} a_n = L$, means that the terms of $\{a_n\}$ become increasingly close to the real number L as n increases without bound. If a limit L exists, the sequence is said to *converge*. If no limit exists, the sequence is said to *diverge*.

48. Thm. **Limit of an Infinite Sequence**: If $\lim_{x\to\infty} f(x) = L$ for a real number L, and $a_n = f(n)$ for every natural number n, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = L$.

49. Def. **Monotonic Sequence**: A sequence $\{a_n\}$ is monotonic if its terms are nondecreasing, $a_1 \le a_2 \le a_3 \le \ldots \le a_n \le \ldots$, or nonincreasing $a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge \ldots$.

50. Def. **Bounded Sequence**: A sequence $\{a_n\}$ is
    a. *bounded from above* if there exists a real number M such that $a_n \le M$ for all n.
    b. *bounded from below* if there exists a real number m such that $m \le a_n$ for all n.
    c. *bounded* if it is both bounded from above and below.

51. Thm. **Convergent Implies Bounded**: Every convergent sequence is bounded.

52. Thm. **Bounded and Monotonic Implies Convergent**: Every bounded monotonic sequence is convergent.

53. Def. **Infinite Series**: An infinite series is the sum of the terms of an infinite sequence. It is denoted $\sum_{n=1}^{\infty} a_n$. Sometimes, it is simply written as $\sum a_n$ where the starting value is assumed to be 1 unless indicated otherwise in the text.

54. Def. **N^{th} Partial Sum**: The n^{th} partial sum of a series $\sum a_n$ is given by $S_n = a_1 + a_2 + a_3 + \ldots + a_n$.

55. Def. **Sum of an Infinite Series**: If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum a_n$ converges, and the limit S is called the sum of the series. If $\{S_n\}$ diverges, then the series diverges.

56. Def. **Geometric Series**: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 \ldots + ar^n + \ldots$, where $a \ne 0$, is called a geometric series.

57. Thm. **Convergence of a Geometric Series**: $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ whenever $|r| < 1$. If $|r| \ge 1$, the series diverges.

58. Def. **Telescoping Series**: A telescoping series is one in which the n^{th} term can be expressed as $a_n = b_n - b_{n+1}$.

59. Thm. ❖**Convergence of a Telescoping Series**: If $\sum a_n$ is a telescoping series with $a_n = b_n - b_{n+1}$, then $\sum a_n$ converges if and only if $\{b_n\}$ converges. Furthermore, if $\{b_n\}$ converges to L, then $\sum a_n = b_1 - L$.

60. Thm. **Properties of Convergent Series**: If $\sum a_n = A$ and $\sum b_n = B$, and c is a real number, then the following series converge to the indicated sums.

    a. $\sum ca_n = c\sum a_n = cA$.

    b. $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$.

    c. CHANGING (altering the value of, deleting, or adding in) a FINITE number of terms in a series does NOT change whether it converges or diverges, although it may change the value of its sum if it does converge.

    d. REGROUPING the values in a series can CHANGE whether it converges or diverges.

61. Thm. ❖***N*****th Term Test for Divergence**: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.
    Restatement: If the n^{th} term of a series does not go to zero, the series diverges.

62. Thm. ❖**Convergence of *P*-Series**: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

    a. converges whenever $p > 1$.

    b. diverges whenever $0 < p \leq 1$.

63. Def. **Harmonic Series**: The p-series for which $p = 1$ is called the harmonic series.

64. Thm. ❖**Integral Test**: If f is *positive*, *continuous*, and *decreasing* for $x \geq 1$ and $a_n = f(n)$, then $\sum_{1}^{\infty} a_n$ and $\int_{1}^{\infty} f(x)dx$ either both converge or both diverge.

65. Thm. ❖**Direct Comparison Test**: Let $\sum a_n$ and $\sum b_n$ be two series of *positive* terms for which $0 < a_n \leq b_n$. Then:

    a. If $\sum b_n$ converges, $\sum a_n$ converges.

    b. If $\sum a_n$ diverges, $\sum b_n$ diverges.

66. Def. **Order of Magnitude of a Series**: Let $\sum a_n$ and $\sum b_n$ be two series of *positive* terms. Then:

    a. $\sum a_n$ has the *same order of magnitude* as $\sum b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where $L > 0$.

    b. $\sum a_n$ has a *lesser order of magnitude* as $\sum b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

    c. $\sum a_n$ has a *greater order of magnitude* as $\sum b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$.

67. Thm. ❖**Limit Comparison Test**: Let $\sum a_n$ and $\sum b_n$ be two series of *positive* terms.

    a. If $\sum a_n$ and $\sum b_n$ are of the same order of magnitude, then either both series converge, or both diverge.

    b. If $\sum a_n$ is of a lesser order of magnitude than $\sum b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

    c. If $\sum a_n$ is of a greater order of magnitude than $\sum b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

68. Def. **Alternating Series**: A series for which the terms are alternately positive and negative is called an alternating series.

69. Def. **Alternating Harmonic Series**: The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ is called the alternating harmonic series.

70. Thm. ❖**Alternating Series Test for Convergence**: Let $a_n > 0$. The alternating series $\sum_{1}^{\infty} (-1)^{n+1} a_n$ converges if both of the following two conditions are met:

    a. $a_{n+1} \le a_n$ and

    b. $\lim_{n\to\infty} a_n = 0$.

71. Thm. ❖**Error Bound for Summing an Alternating Series**: Let $\sum_{1}^{\infty} (-1)^{n+1} a_n$ be a convergent alternating series for which $a_n \ge a_{n+1}$. Then the absolute value of the error R_n in approximating the sum S with the n^{th} partial sum S_n, is $|R_n| = |S - S_n| \le a_{n+1}$.

72. Def. **Absolutely Convergent Series**: The series $\sum a_n$ is called absolutely convergent if both $\sum a_n$ and $\sum |a_n|$ converge.

73. Def. **Conditionally Convergent Series**: The series $\sum a_n$ is called conditionally convergent if $\sum a_n$ converges and $\sum |a_n|$ diverges.

    WARNING: Rearranging the terms of a *conditionally* convergent series can change the sum of the series. It could also cause it to diverge.

74. Thm. **The Absolute Convergence Theorem**: If $\sum |a_n|$ converges, then $\sum a_n$ converges.

    Note that the contrapositive of this statement must therefore be true: If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

75. Thm. ❖**Ratio Test**: Let $\sum a_n$ be a series of nonzero terms.

    a. If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$, the series converges absolutely.

    b. If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$ (or infinite), the series diverges.

    c. If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive.

76. Process **Analyzing an Infinite Series:** To determine if a given series diverges or converges:

    a. Determine whether or not the limit of the n^{th} term is zero.

    b. Determine if the series is a special type: *p*-series, geometric series, telescoping series, or alternating series.

    c. Determine if the Ratio Test or Integral Test can be applied successfully.

    d. Determine if a Direct or Limit Comparison Test can be applied successfully.

---*End for Chapter 9 Quiz*---

77. Thm. ❖**Taylor Polynomial of Degree *n* Approximating $f(x)$ for x near c:** If $f(x)$ is a differentiable function through order n in an interval containing c, then $f(x)$ can be approximated by the polynomial

    $P_n(x) = f(c) + f'(c)(x-c) + \dfrac{f''(c)}{2!}(x-c)^2 + \dfrac{f'''(c)}{3!}(x-c)^3 + \ldots + \dfrac{f^{(n)}(c)}{n!}(x-c)^n$. We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = c$. It is called the n^{th} degree Maclaurin polynomial for f when $c = 0$.

78. Thm. **Taylor's Theorem:** If $f(x)$ is a differentiable function through order $n+1$ on an interval I containing c, then for each x in I there exists a value z between x and c such that $f(x)$ can be expressed as $f(x) = P_n(x) + R_n$ where $R_n = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$.

79. Cor. ❖**Lagrange Error Bound for $P_n(x)$**: The maximum error R_n in approximating a function $f(x)$ with a Taylor polynomial of degree n can be calculated by $|R_n| \le \dfrac{M}{(n+1)!}|x-c|^{n+1}$ where M is the maximum value of $\left|f^{(n+1)}\right|$ on the interval between x and c.

80. Def. **Power Series**: For a variable x and a constant c, a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$ is called a power series centered at c. (Note that on the left side when $x = c$, we have a factor of 0^0 in the first term, which is normally considered undefined. In this context we agree that the first term of the series is always a_0, even if $x = c$.)

81. Thm. **Convergence of a Power Series**: For a power series centered at c, precisely one of the following is true:

   a. The series converges only at the point c.

   b. There exists a real number $R > 0$ such that the series converges *absolutely* for $|x-c| < R$, and diverges for $|x-c| > R$.

   c. The series converges *absolutely* for all real numbers x.

82. Def. **Radius of Convergence**: The number R in the above theorem is called the radius of convergence. If the series converges at only one point, we define R to be 0. If the series converges absolutely for all real numbers, we consider R to be ∞.

83. Def. **Domain of a Power Series (or Interval of Convergence)**: If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, then the domain of the function is the set of all values of x for which the series converges. The domain of the power series is also called the *interval of convergence*.

84. Thm. **Properties of Functions Defined by Power Series**: For a powers series given by $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ with a radius of convergence $R > 0$, then on the interval $(c-R, c+R)$, f is differentiable (and therefore continuous and integrable). Moreover,

   a. $f'(x) = \sum_{n=0}^{\infty} n \bullet a_n(x-c)^{n-1}$ and R is its radius of convergence.

   b. $\int f(x)dx = C + \sum_{n=0}^{\infty} \dfrac{a_n}{n+1}(x-c)^{n+1}$ and R is its radius of convergence.

NOTE: Though the radius of convergence is unaltered by differentiation or integration, the endpoints on the interval of convergence may change. In particular, differentiation may result in the loss (never gain) of an endpoint and integration may result in the gain (never loss) of an endpoint.

85. Def. **Geometric Power Series**: A function of the form $f(x)=\dfrac{a}{1-x}$ can be expressed as a power series centered at $c=0$ with radius of convergence $R=1$ using a geometric series of the form $f(x)=\dfrac{a}{1-x}=a+ax+ax^2+ax^3+\dots$.

86. Def. ❖**Taylor and Maclaurin Series:** If $f(x)$ has derivatives of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\dots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\dots$$

is called the Taylor series for $f(x)$ at c. In addition, if $c=0$, then the series is called the Maclaurin series for f.

87. Thm. **Convergence of a Taylor Series:** If $\lim_{n\to\infty} R_n = 0$ (where R_n is the error expression defined in Taylor's Theorem) for all x in the interval I, then the Taylor series for f converges and equals $f(x)$. That is $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$.

88. Thm. **Uniqueness of the Power Series**: The power series representation of a function $f(x)$ about $x=c$ is unique.

89. Thm. ❖**Power Series for Elementary Functions**:

a. $e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dots+\dfrac{x^n}{n!}+\dots$ where $I=(-\infty,\infty)$

b. $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dots+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+\dots$ where $I=(-\infty,\infty)$

c. $\cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\dots$ where $I=(-\infty,\infty)$

d. $\dfrac{1}{1-x}=1+x+x^2+x^3+\dots+x^n+\dots$ where $I=(-1,1)$

--*End for Chapter 9 Test*--