

## DEFINITIONS, THEOREMS, AND FORMULAS – Semester 1

### Chapter 4 – Integration

❖ *Indicates that the item should be memorized in exact detail. You may be asked to quote it on a quiz or test.*

1. Thm. **Area Existence:** If  $f$  is a continuous function on  $[a, b]$ , then the limits as  $n \rightarrow \infty$  of both the lower and upper sums exist and are equal to each other. That is

$$\lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n f(m_i) \Delta x \right] = \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n f(M_i) \Delta x \right] \text{ where } \Delta x = \frac{b-a}{n} \text{ and } f(m_i) \text{ and } f(M_i)$$

represent the minimum and maximum values, respectively, of  $f$  on the interval.

2. Def. **Area Under a Curve:** Let  $f$  be a continuous function on the interval  $[a, b]$ . The area of the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$  is:

$$\text{area} = \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n |f(c_i)| \cdot \Delta x \right] \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_{i-1} \leq c_i \leq x_i.$$

3. Thm. **Properties of Summations:**

a. 
$$\sum_{i=1}^n c = cn$$

b. 
$$\sum_{i=1}^n c \cdot f(i) = c \cdot \sum_{i=1}^n f(i)$$

c. 
$$\sum_{i=1}^n [f(i) \pm g(i)] = \sum_{i=1}^n f(i) \pm \sum_{i=1}^n g(i)$$

4. Def. **Riemann Sum:** Let  $f$  be defined on the closed interval  $[a, b]$  which is partitioned by the set  $\{a = x_0, x_1, x_2, \dots, x_n = b\}$ . If  $c_i \in [x_{i-1}, x_i]$  and  $\Delta x_i = x_i - x_{i-1}$ , then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i \text{ is called a Riemann sum of } f \text{ for the given partition.}$$

5. Def. **Norm of the Partition:** The length of the largest subinterval of a partition is called the norm of the partition and is denoted by  $\|\Delta\|$ .
6. Def. **Regular Partition:** If every subinterval in a partition is of equal length, the partition is called regular.

7. Def. **❖Definite Integral:** If  $f$  is defined on the interval  $[a, b]$  and the limit of the Riemann sum

$$\lim_{\|\Delta\| \rightarrow 0} \left[ \sum_{i=1}^n f(c_i) \Delta x_i \right] \text{ exists, then this limit is called the definite integral of } f \text{ on } [a, b] \text{ and}$$

is denoted by  $\int_a^b f(x) dx = \lim_{\|\Delta\| \rightarrow 0} \left[ \sum_{i=1}^n f(c_i) \Delta x_i \right]$ . (The values  $a$  and  $b$  are called the lower and upper limits of the integral, respectively.)

8. Def. **Integrable Function:** A function is said to be integrable on an interval if it has a definite integral on the interval.

9. Thm. **Continuity and Integrability:** If a function  $f$  is continuous on the closed interval  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .

10. Thm. **Area Under a Curve:** Let  $f$  be a continuous, function on the interval  $[a, b]$ . The area of the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$  is found by:

$$\text{area} = \int_a^b |f(x)| dx .$$

11. Def. **Definite Integral at a Point:**  $\int_a^a f(x) dx = 0$ .

12. Def. **Definite Integral over a Reverse Interval:** If  $f$  is an integrable function on  $[a, b]$ , then

$$\int_b^a f(x) dx = - \int_a^b f(x) dx .$$

13. Thm. **Definite Integral over Two Adjacent Intervals:** If  $f$  is an integrable function on the three

intervals indicated, then  $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$

14. Thm. **Definite Integral of a Constant Times a Function:** If  $f$  is an integrable function on  $[a, b]$

and  $k$  is a constant, then  $\int_a^b k \cdot f(x) dx = k \cdot \int_a^b f(x) dx$ .

15. Thm. **Definite Integral of a Sum or Difference of Functions:** If  $f$  and  $g$  are both integrable

functions on  $[a, b]$ , then  $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$ .

16. Thm. **Comparing Definite Integrals:** If  $f$  and  $g$  are both integrable functions on  $[a, b]$  and

$f(x) \leq g(x)$  for  $a \leq x \leq b$ , then  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ .

17. Thm. **Integration of an Odd Function:** If  $f$  is an odd function which is integrable on  $[-a, a]$ ,

$$\text{then } \int_{-a}^a f(x)dx = 0.$$

18. Thm. **Integration of an Even Function:** If  $f$  is an even function which is integrable on  $[-a, a]$ ,

$$\text{then } \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

19. Thm. **Distance versus Displacement:** If the continuous function  $v(t)$  represents the velocity of a function over an interval of time  $[a, b]$ , then:

$$\text{Displacement (or change in position)} = \int_a^b v(t)dt.$$

$$\text{Total Distance Traveled} = \int_a^b |v(t)|dt.$$

20. Thm. **❖Fundamental Theorem of Calculus:** If a function  $f$  has a continuous derivative on the

$$\text{interval } [a, b], \text{ then } f(b) - f(a) = \int_a^b f'(x)dx.$$

**Restatement:** The total change in a function is the definite integral of its rate of change.

21. Def. **Average Value of a Function:** If  $f$  is a continuous function on  $[a, b]$ , then the average

$$\text{value of } f \text{ on } [a, b] \text{ is found by } \bar{y} = \frac{1}{b-a} \int_a^b f(x)dx.$$

22. Thm. **❖Mean Value Theorem for Integrals:** If  $f(x)$  is continuous on  $[a, b]$ , then there exists

$$\text{a number } c \text{ in } (a, b) \text{ such that } f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

**Restatement:** Under the given conditions, there is a point in the open interval where the value of the function is equal to the average value of the function over the interval.

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the value of the function corresponds to the height of a rectangle, with base  $(b-a)$ , whose area is the same as the area under the curve between the two endpoints.

23. Def. **Antiderivative:** If  $f'(x)$  is the derivative of  $f(x)$ , then  $f(x)$  is called an antiderivative of  $f'(x)$ .

24. Thm. **Relationship Between Antiderivatives:** If  $F$  and  $G$  are both antiderivatives of a function  $f$ , then  $F(x) = G(x) + C$  for some constant  $C$ .

25. Def. **Accumulation Function:** Let  $f$  be continuous on  $[a, b]$  and  $x \in [a, b]$ . The function

$$A(x) = \int_a^x f(t) dt \text{ is called the accumulation function.}$$

26. Thm. **❖Second Fundamental Theorem of Calculus:** If  $f$  is continuous on an open interval containing  $a$ , then for every  $x$  in the interval where  $F(x) = \int_a^x f(t) dt$ ,  $F(x)$  is an antiderivative of  $f(x)$ .

-----End for Chapter 4 Quiz-----

27. Def. **❖Indefinite Integral:** The expression  $\int f(x) dx$  is called the indefinite integral and is used to represent the family of all antiderivatives of  $f(x)$ .

28. Thm. **Properties of the Indefinite Integral:**

a.  $\int dx = x + C .$

b.  $\int k \cdot f(x) dx = k \cdot \int f(x) dx .$

c.  $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx .$

29. Thm. **Trapezoidal Rule:** If  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x) dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

30. Thm. **Simpson's Rule:** If  $f$  is continuous on  $[a, b]$  and  $T$  represents the approximation of

$$\int_a^b f(x) dx \text{ using the Trapezoidal Rule and } M \text{ represents the same integral approximated}$$

with the same number of subdivisions using the Midpoint Rule, then the value  $S = \frac{2M + T}{3}$  is the Simpson's Rule approximation for the integral.

-----End for Chapter 4 Test-----

## Chapter 5 – Transcendental Functions

31. Def. **e**:  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$  or  $e = \lim_{n \rightarrow 0} (1+n)^{\frac{1}{n}}$ .
32. Def. **Exponential Function**:  $y = a^x$  where  $a > 0$  and  $a \neq 1$  is called the exponential function.
33. Def. **Logarithm Function**: If  $y = \log_a x$ , then  $a^y = x$ , where  $a > 0$ ,  $a \neq 1$ , and  $x > 0$ .  
Notation:  $\ln x = \log_e x$  and  $\log x = \log_{10} x$ .
34. Thm. **Properties of Logarithms**:
- a.  $\log_b 1 = 0$ .
  - b.  $\log_b pq = \log_b p + \log_b q$ .
  - c.  $\log_b \frac{p}{q} = \log_b p - \log_b q$ .
  - d.  $\log_b p^n = n \log_b p$ .
  - e.  $\log_a x = \frac{\log_b x}{\log_b a}$ .
  - f.  $b^{\log_b x} = x$ .
  - g.  $\log_b b^x = x$ .
35. Thm. **Properties of the Natural Logarithm**.
- a. The domain of  $y = \ln x$  is  $(0, \infty)$  and the range is  $(-\infty, \infty)$ .
  - b. The function is continuous, increasing, one-to-one.
  - c. The graph is concave downward on its entire domain.
  - d.  $\lim_{x \rightarrow \infty} \ln x = \infty$  and  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .
36. Thm. **Derivative of the Natural Logarithm**. If  $y = \ln x$ , then  $y' = \frac{1}{x}$ .
37. Thm. **Derivative of the Natural Logarithm with Absolute Value**: If  $y = \ln|x|$ , then  $y' = \frac{1}{x}$ .
38. Thm. **Integration of  $\frac{1}{x}$** :  $\int \frac{1}{x} dx = \ln|x| + C$  for  $x \neq 0$ .
39. Thm. **Integral of the Sine Function**:  $\int \sin x dx = -\cos x + C$ .
40. Thm. **Integral of the Cosine Function**:  $\int \cos x dx = \sin x + C$ .

41. Thm. **Integral of the Tangent Function:**  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C .$
42. Thm. **Integral of the Cotangent Function:**  $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C .$
43. Thm. **Integral of the Secant Function:**  $\int \sec x dx = \ln|\sec x + \tan x| + C .$
44. Thm. **Integral of the Cosecant Function:**  $\int \csc x dx = -\ln|\csc x + \cot x| + C .$
45. Def. **Inverse Function:** A function  $g$  is the inverse of the function  $f$  if  $f(g(x)) = x$  for each  $x$  in the domain of  $g$  and  $g(f(x)) = x$  for each  $x$  in the domain of  $f$ . Notation:  
 $f^{-1}(x) = g(x) .$
46. Thm. **Basic Properties of Inverse Functions:**
- If  $(a, b)$  is a point on  $f(x)$ , then  $(b, a)$  is a point on  $f^{-1}(x)$ .
  - The domain of  $f$  is the range of  $f^{-1}$  and the range of  $f$  is the domain of  $f^{-1}$ .
  - The graphs of  $f$  and  $f^{-1}$  are mirror reflections about the diagonal line  $y = x$ .
47. Def. **One-to-One:** A function is called one to one if no two ordered pairs have the same first or second member.
48. Thm. **Existence of an Inverse:** A function possess an inverse if and only if it is one-to-one.
49. Def. **Strictly Monotonic:** A function is strictly monotonic if it is either increasing on its entire domain or decreasing on its entire domain.
50. Thm. **Monotonic Implies 1:1:** If a function is strictly monotonic on an interval, then it is one-to-one on the interval and has an inverse.
51. Thm. **Advanced Properties of Inverse Functions:**
- If  $f$  is continuous, then  $f^{-1}$  is continuous.
  - If  $f$  is increasing, then  $f^{-1}$  is increasing.
  - If  $f$  is decreasing, then  $f^{-1}$  is decreasing.
  - If  $f$  is differentiable at  $x = c$  and  $f'(c) \neq 0$ , then  $f^{-1}$  is differentiable at  $f(c)$ .
52. Thm. **Derivative of an Inverse Function:** If  $f$  is a differentiable function that possesses an inverse function  $g$ , then  $g'(x) = \frac{1}{f'(g(x))} .$
53. **Restatement:** If  $f$  and  $g$  are inverse functions, and  $(a, b)$  is a point on the function  $g$ , then  
 $g'(a) = \frac{1}{f'(b)} .$

54. Thm. **Properties of the Natural Exponential Function:**

- The domain of  $y = e^x$  is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .
- The function is continuous, increasing, one-to-one.
- The graph is concave upward on its entire domain.
- $\lim_{x \rightarrow -\infty} e^x = 0$  and  $\lim_{x \rightarrow +\infty} e^x = \infty$ .

55. Thm. **Derivative of the Natural Exponential Function:** If  $y = e^x$ , then  $y' = e^x$ .56. Thm. **Integration of the Natural Exponential Function:**  $\int e^x dx = e^x + C$ .57. Thm. **Derivative of the General Exponential Function:** If  $y = a^x$ , then  $y' = a^x \ln a$ .58. Thm. **Integration of the General Exponential Function:**  $\int a^x dx = \frac{1}{\ln a} a^x + C$ .59. Thm. **Derivative of the General Logarithm Function:** If  $y = \log_a x$ , then  $y' = \frac{1}{x \ln a}$ .

-----End for Chapter 5 Quiz-----

60. Def. **Inverse Sine:**  $y = \arcsin x$  means  $x = \sin y$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .61. Def. **Inverse Cosine:**  $y = \arccos x$  means  $x = \cos y$  where  $0 \leq y \leq \pi$ .62. Def. **Inverse Tangent:**  $y = \arctan x$  means  $x = \tan y$  where  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .63. Def. **Inverse Cotangent:**  $y = \text{arccot } x$  means  $x = \cot y$  where  $0 < y < \pi$ .64. Def. **Inverse Secant:**  $y = \text{arcsec } x$  means  $x = \sec y$  where  $0 \leq y \leq \pi$  and  $y \neq \frac{\pi}{2}$ .65. Def. **Inverse Cosecant:**  $y = \text{arccsc } x$  means  $x = \csc y$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$  and  $y \neq 0$ .66. Thm. **Derivative of the Inverse Sine:** If  $y = \arcsin x$ , then  $y' = \frac{1}{\sqrt{1-x^2}}$ .67. Thm. **Derivative of the Inverse Cosine:** If  $y = \arccos x$ , then  $y' = \frac{-1}{\sqrt{1-x^2}}$ .68. Thm. **Derivative of the Inverse Tangent:** If  $y = \arctan x$ , then  $y' = \frac{1}{1+x^2}$ .69. Thm. **Derivative of the Inverse Cotangent:** If  $y = \text{arccot } x$ , then  $y' = \frac{-1}{1+x^2}$ .

70. Thm. **Derivative of the Inverse Secant:** If  $y = \operatorname{arcsec} x$ , then  $y' = \frac{1}{|x|\sqrt{x^2 - 1}}$ .
71. Thm. **Derivative of the Inverse Cosecant:** If  $y = \operatorname{arccsc} x$ , then  $y' = \frac{-1}{|x|\sqrt{x^2 - 1}}$ .
72. Thm. **Integration with the Inverse Sine Function:**  $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ .
73. Thm. **Integration with the Inverse Tangent Function:**  $\int \frac{1}{1+x^2} dx = \arctan x + C$ .
74. Thm. **Integration with the Inverse Secant Function:**  $\int \frac{1}{x\sqrt{x^2 - 1}} dx = \operatorname{arcsec}|x| + C$ .

-----End for Chapter 5 Test-----

## Chapter 6 – Differential Equations

75. Def. **Differential Equation:** A differential equation in  $x$  and  $y$  is an equation that involves  $x$ ,  $y$ , and derivatives of  $y$ .
76. Def. **Solution of a Differential Equation:** A solution of a differential equation is a function or relation in  $x$  and  $y$  that is *defined on an open interval* and makes the differential equation true.
77. Thm. **❖Euler's Method:** For a differentiable relation in  $y$  and  $x$  where  $y'$  is known and  $y_0$  is known, then any value of  $y$  can be approximated by the iterative procedure  $y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$ .
78. Def. **Stable Equilibrium:** A relationship where the dependent variable, regardless of its initial value, tends toward a constant state as the independent variable tends to infinity, is called a stable equilibrium.
79. Def. **Equilibrium Solution:** The constant value to which the dependent variable approaches in a stable equilibrium is called the equilibrium solution.
80. Thm. **Stable Equilibriums:** Differential equations of the form  $\frac{dy}{dt} = -k(y - E)$ , where  $k > 0$ , are stable equilibriums.
81. Thm. **Equilibrium Solution:** The value for  $y$  which makes the derivative zero is the equilibrium solution.
82. Thm. **Newton's Law of Cooling:** The rate at which a body cools is proportional to the difference between its temperature and that of the surrounding air.
83. Thm. **Removal of Pollutants:** (You are not required to learn this theorem.) If a lake of volume  $V$  with water flow  $r$  contains an amount  $Q$  of pollutants, then the rate of change of the

- pollutants with respect to time  $t$  can be described as  $\frac{dQ}{dt} = \frac{-r}{V}Q$ , presuming no additional pollutants enter the lake.
84. Def. **Cooling Curve:** A cooling curve is a curve whose differential equation is of the form  $\frac{dy}{dt} = -k(y - E)$ , where  $k$  is positive and  $y_0 > E$ .
85. Thm. **Characteristics of the Cooling Curve:** The cooling curve:
- Has a growth rate that is always negative.
  - Decreases rapidly at the start and more slowly as time passes.
  - Is a stable equilibrium, with equilibrium solution  $y = E$ .
86. Def. **Learning Curve:** A learning curve is a curve whose differential equation is of the form  $\frac{dy}{dt} = -k(y - E)$ , where  $k$  is positive and  $y_0 < E$ .
87. Thm. **Characteristics of the Learning Curve:** The learning curve:
- Has a rate of growth that is always positive.
  - Increases rapidly at the start and more slowly as time passes.
  - Is a stable equilibrium, with equilibrium solution  $y = E$ .
88. Def. **Logistic Curve:** A logistic curve is a curve whose differential equation is of the form  $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$  where  $k > 0$ .  $L$  is called the *carrying capacity* of the environment, or the *limiting value* of the population.
89. Thm. **Logistic Curve:** For values of  $y$  where  $0 < y < L$ , the logistic curve:
- Has a growth rate that is always positive.
  - Increases slowly at the start, rapidly later on, and slowly again as the limiting value  $L$  is reached.
  - Has its fastest rate of growth when  $y = \frac{L}{2}$ .
  - Has a stable equilibrium solution of  $y = L$ .
  - Has a relative growth rate (growth rate as a percent of the whole population) that is a linearly decreasing function of  $y$ .

-----End for Chapter 6 Quiz-----

90. Thm. **Partial Fractions with Non-Repeated Linear Factors:** A fraction of the form  $\frac{P(x)}{(ax+b)(cx+d)}$  where  $P(x)$  is a polynomial of degree less than the denominator can always be rewritten in the form  $\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$  where  $A$  and  $B$  are constants. This statement can be generalized for a fraction with  $n$  linear factors in the denominator.

91. Thm. **Partial Fractions with Repeated Linear Factors:** A fraction of the form  $\frac{P(x)}{(ax+b)^3}$  where  $P(x)$  is a polynomial of degree less than the denominator can always be rewritten in the form  $\frac{P(x)}{(ax+b)^3} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3}$  where  $A$ ,  $B$ , and  $C$  are fractions. This statement can be generalized for a fraction with  $n$  repeated linear factors in the denominator.

## Chapter 7 – Applications of Integration

92. Thm. **Area Between Two Curves:** If  $f$  and  $g$  are continuous on  $[a, b]$  and  $g(x) \leq f(x)$  for all  $x$  in  $[a, b]$ , then the area of the region bounded by the graphs of  $f$  and  $g$  and the vertical lines  $x = a$  and  $x = b$  is  $\int_a^b f(x) - g(x) dx$ .

-----End for Chapter 6 Test-----

93. Def. **Lorenz Curve:** The Lorenz curve indicates the total income which is received by the bottom  $t$  proportion ( $0 \leq t \leq 1$ ) of the population.
94. Def. **Gini Index:** The Gini Index is twice the area between the Lorenz curve and the Egalitarian Line.

95. Thm. **Gini Index:** Let  $L(x)$  represent the Lorenz curve. Then  $\text{Gini Index} = 2 \int_0^1 [x - L(x)] dx$ .

-----End for Chapter 7 Quiz-----

96. Def. **Smooth Curve:** A curve defined by  $y = f(x)$  on the interval  $[a, b]$  is called smooth if  $f'(x)$  is continuous on  $[a, b]$ .
97. Def. **Arc Length:** Let  $y = f(x)$  be a smooth curve on the interval  $[a, b]$  then the arc length of  $f$  between  $a$  and  $b$  is given by  $s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$ . Similarly, for a smooth curve given by  $x = g(y)$ , the arc length of  $g$  between  $c$  and  $d$  is  $s = \int_c^d \sqrt{1 + [g'(y)]^2} dy$ .

98. Def. **Work:** If a constant force  $F$  acts over a fixed distance  $D$ , then the work done is  $W = FD$ .
99. Def. **Pressure:** Pressure is the force *per* unit area on the surface of a body.
100. Thm. **Fluid Pressure:** Fluid Pressure  $P$  on an object is the density  $D$  of the fluid times the depth  $h$  of the object in the liquid.  $P = Dh$ .
101. Thm. **Fluid Force:** The fluid force  $FF$  on a submerged *horizontal* surface of area  $A$  is the pressure on the surface times the area of the surface.  $FF = PA$ .

-----End for Chapter 7 Test-----