CLASSWORK E – Beyond the AP

Volume of a Solid of Revolution – Shell Method

Lesson Goal: Find the volume of a solid of revolution by the shell method.

1. Given a region bounded by the curve $y = x^2$, $y = 0$, and $x = 3$. Find the volume of the solid formed by revolving the region about the y-axis.

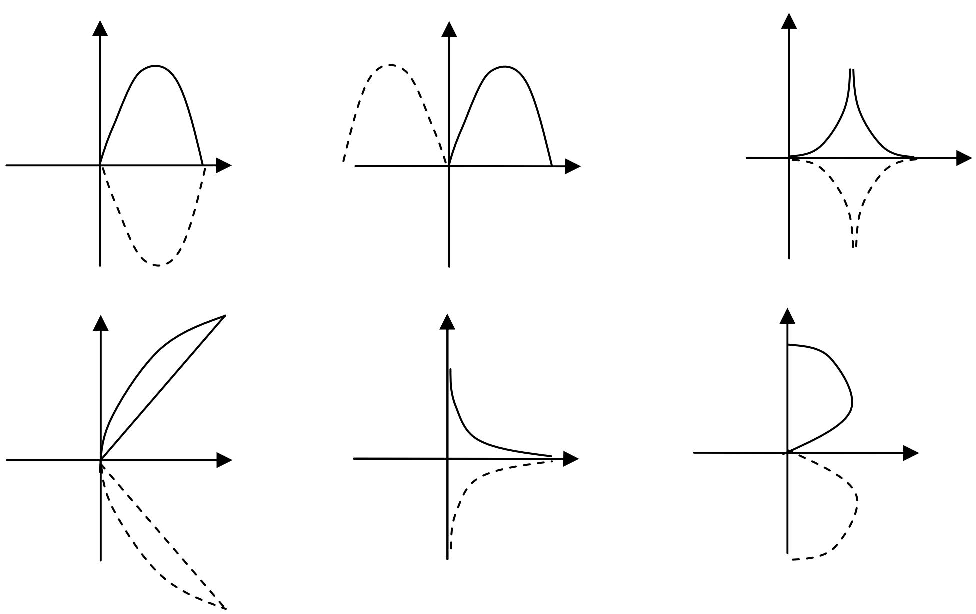
2. Given a region bounded by the curve $y^2 = 12 - 4x$, $y = 0$, $x = 0$, and $y = 2$. Find the volume of the solid formed by revolving the region about the x-axis.

3. Let R be the region enclosed by the graphs of $y = e^x$, and $y = (x+1)^2$. Find the volume of the solid generated when R is revolved about the y-axis.

4. Given the region bounded by the curves $y = \ln x$, $y = \frac{x}{4}$, and the x-axis. Find the volume of the solid formed when the region is revolved about the x-axis.

Lesson Goal: Determine when the shell method is preferable to the disk method.

5. Choose the Best Method for Finding the Volume of the Solid of Revolution



6. Given a region in the *second* quadrant bounded by the curve $y = x^2$, $y = \frac{-x}{2} + 3$, and $x = 0$. Find the volume of the solid formed by revolving the region about the line $x = 1$.

7. Given a region bounded by the curve $x - y = -1$, $y = 0$, and $x = 0$. Find the volume of the solid formed by revolving the region about the line $y = -1$.

8. Given a region bounded by the curve $y = \dfrac{1}{\sqrt{x}}$, $y = 0$, and the lines $x = 4$ and $x = 9$. Find the volume of the solid formed by revolving the region about the line $x = 10$.

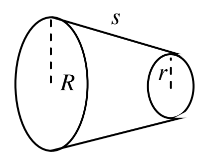
9. Given a region bounded by the curve $y = \dfrac{4}{x+1}$, $x = 3$, and the *x*-axis, and the *y*-axis. Find the volume of the solid formed by revolving the region about the line $x = -1$.

Lateral Surface Area on a Solid of Revolution

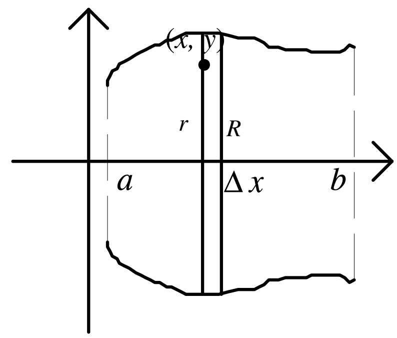
Lesson Goal: Find the lateral surface area on a solid of revolution.

10. How would we approximate the area of a surface of revolution?

At right is illustrated a frustum of a cone. The surface area of a frustum of a cone is found by the formula $\pi(R+r)s$



11. Suppose that we slice a surface of revolution into circular slices, just as we would slice a solid of revolution.



   a. The surface area of each subdivision approximates that of the lateral area of a frustum of a cone. Hence we use the formula:

   $\Delta SA \approx \pi(R+r)\Delta s$

   b. Since the two radii are almost the same (for thin slices) we will use the same value for each.

   $\Delta SA \approx \pi(2 \bullet r)\Delta s$

   c. Since Δs corresponds to the length of the arc in the subdivision, by substitution (using the arc length formula in the previous lesson) we have:

   $\Delta SA \approx$ ________________________________

12. To get an exact value for the surface area, we take the limit of the summation as the norm of the partition approaches zero.

   $SA =$ ___________________________________

   To emphasize that the radius must be expressed as a function of x, we normally write the formula as:

   $$SA = 2\pi \int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$$

13. Find the area of the surface obtained by revolving $x^2 + y^2 = 4$ about the x-axis from $\frac{1}{2} \le x \le 1$.

14. Find the area of the surface obtained by revolving $y = x^2$ about the *y*-axis from $0 \le x \le 1$.

15. Find the area of the surface obtained by revolving $x = \frac{1}{4}y^2$ about the *x*-axis from $1 \le x \le 4$. Use TI-CAS to evaluate.

Trigonometric Substitutions

Lesson Goal: Use a sine or tangent substitution to evaluate an integral.

16. Compare: $\int x\sqrt{4-x^2}\,dx$, $\int x^2\sqrt{4-x^2}\,dx$, $\int x^3\sqrt{4-x^2}\,dx$

    a. Which two of these do you already know how to integrate?

    b. Why do those techniques not work on the other one?

Integrate:

17. $\int x^2\sqrt{4-x^2}\,dx =$

18. $\int \frac{dx}{\left(x^2+1\right)^{\frac{3}{2}}} =$

19. $\int \frac{\sqrt{25-x^2}}{x} dx =$

20. $\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}} =$

Lesson Goal: Use a secant substitution to evaluate an integral.

21. $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x^2}\,dx =$

22. $\displaystyle\int \frac{dx}{\left(4x^2-9\right)^{\frac{3}{2}}} =$

23. $\displaystyle\int \frac{dx}{\left(x^2+1\right)^2} =$

24. $\displaystyle\int \frac{\sqrt{4-3x^2}}{x^4}\,dx =$

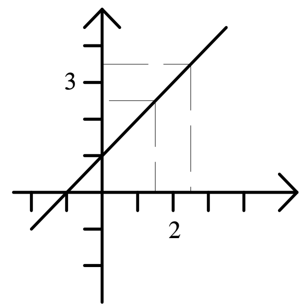
Two-Sided Limits

Lesson Goal: Prove the existence of a two-sided limit using the definition.

Definition: **Two-sided limit at a point**: Let f be a function defined on an open interval containing a (except possibly at a) and let L be a real number. $\lim_{x\to a} f(x) = L$ means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Restatement: The limit of $f(x)$ as x approaches a is L means that the following must be true. For every ε-neighborhood around L there exists a δ-neighborhood around a so that for every x in the δ-neighborhood, $f(x)$ is in the ε-neighborhood.

25. Consider: $\lim_{x\to 2} x+1=3$.



a. For $\varepsilon = \frac{1}{2}$, find δ.

b. For $\varepsilon = \frac{1}{4}$, find δ.

Setting the window size on our TI-CAS to match the epsilon and delta neighborhoods, we can perform a similar illustration.

26. Verify that $\lim_{x\to 3}(2x-1)=5$.

a. Set the vertical windows dimensions to $-4.5 \le y \le 5.5$ and find horizontal window dimensions so that the viewing rectangle captures the complete graph of the function over the stated domain. Start with $2 \le x \le 4$. It fails to capture the graph. Retry with $2.5 \le x \le 3.5$. It also fails to capture the function. Retry with $2.8 \le x \le 3.2$. That captures the entire function.

b. Change the window dimensions to $-4.8 \le y \le 5.2$. Now find a neighborhood for x that captures the function.

27. Verify that $\lim_{x\to 3}(2x-1) \ne 4$.

a. Choose window dimensions of $2 \le y \le 6$. Find dimensions for x that capture the function.

b. Change the window dimensions to $3.5 \le y \le 4.5$. Now try to find dimensions for x that capture the function.

28. Prove: $\lim_{x \to 3} (2x - 1) = 5$

29. Prove: $\lim_{x \to 1} (3x + 4) = 7$

30. Prove: $\lim_{x \to 4} x^2 = 16$

Infinite Limits

Lesson Goal: Prove that a limit is infinite.

Definition: **Limit of positive infinity**: $\lim_{x\to a} f(x) = +\infty$ means that for every $N > 0$ there exists a $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $f(x) > N$.

Definition: **Limit of negative infinity**: $\lim_{x\to a} f(x) = -\infty$ means that for every $N < 0$ there exists a $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $f(x) < N$.

31. Illustration: Consider $\lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$.

a. For $N = 20$, find δ.

b. For $N = 100$, find δ.

32. Prove: $\lim_{x\to -4^+} \dfrac{1}{x+4} = +\infty$

33. Prove: $\lim_{x\to 2} \dfrac{-1}{(x-2)^2} = -\infty$

34. Prove: $\lim_{x \to 5} \frac{x-8}{(x-5)^2} = -\infty$

Negation of a Limit

Lesson Goal: Prove that a two-sided limit does not exist and prove that a limit exists at infinity.

35. What does it mean to say that a limit does NOT exist?

Review: **Two-sided limit at a point**: Let f be a function defined on an open interval containing a (except possibly at a) and let L be a real number. $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

36. Below are listed six statements which attempt to state the negation of the above definition. Only two are correct negations. Can you identify the correct two?

a. For every $\varepsilon > 0$ there does not exist a $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

b. For at least one $\varepsilon > 0$ there does not exist any $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

c. For at least one $\varepsilon > 0$ and every $\delta > 0$ there exists an x such that $0 < |x-a| < \delta$ and $|f(x) - L| > \varepsilon$.

d. For at least one $\varepsilon > 0$ there exists a $\delta > 0$ and an x such that $0 < |x-a| < \delta$ and $|f(x) - L| > \varepsilon$.

e. For at least one $\varepsilon > 0$ and every $\delta > 0$, whenever $0 < |x-a| < \delta$, then $|f(x) - L| > \varepsilon$.

f. For every $\varepsilon > 0$ there does not exist $\delta > 0$ such that whenever $0 < |x-a| < \delta$, then $|f(x) - L| > \varepsilon$.

The model for writing negation proofs is letter ___________

37. Prove: $\lim_{x \to 3} f(x) \neq -2$ where . $f(x) = \begin{cases} -x+1 & x \leq 3 \\ x+2 & x > 3 \end{cases}$

Limits at Infinity

Definition: **Limit at positive infinity**: $\lim_{x \to +\infty} f(x) = L$ means that for every $\varepsilon > 0$ there exists an $M > 0$ such that whenever $x > M$, then $|f(x) - L| < \varepsilon$.

Deinition: **Limit at negative infinity**: $\lim_{x \to -\infty} f(x) = L$ means that for every $\varepsilon > 0$ there exists an $M < 0$ such that whenever $x < M$, then $|f(x) - L| < \varepsilon$.

38. Prove: $\lim_{x \to +\infty} \frac{x-3}{x} = 1$

39. Prove: $\lim_{x \to -\infty} \frac{2x+5}{x} = 2$