CLASSWORK 9 – Infinite Series

Lesson Goal: Determine if an infinite sequence is bounded, monotonic and/or convergent.

To calculate the limit of a sequence, we use everything that we know about the limits of functions, because *sequences are functions*. Specifically that would include:

- Sum, difference, product, and quotient properties
- Squeeze Law
- L'Hôpital's Rule

Determine the convergence or divergence of the sequence.

1. $a_n = \dfrac{2n+1}{n}$

2. $a_n = 1 - \dfrac{1}{2^n}$

3. $a_n = \dfrac{n}{e^n}$

4. $a_n = n^{-1.6} + e^n$

5. $a_n = 1 + (-1)^n$

6. Graph $a_n = \dfrac{n!}{3^n}$ and determine if it converges.

   Clear any existing graphs by changing $f1$ to zero.

   Go to the graphing window and press (*menu*) *Graph Type* *Sequence* *Sequence* (*enter*)

   Tab to get to the entry screen. Beside $u1(n)$ type $\dfrac{n!}{3^n}$

   (Note that the variable must be *n*.)

   Change the number of terms to $1 \le n \le 8$

   (*menu*) *Window* *Zoom – Fit* (*enter*)

Determine whether the sequence is monotonic.

7. $a_n = \sin(n)$

8. $a_n = \sin(2\pi n)$

9. $a_n = 2n + \sin(n)$

Comparing the above two problems demonstrates an important point. A monotonic function will guarantee that a related sequence is monotonic. However, the converse is not true.

Determine whether the sequence is bounded.

10. $a_n = 2 + \frac{1}{n}$

11. $a_n = \left(\frac{-1}{5}\right)^n$

12. $a_n = \frac{n^3}{2n - n^2}$

Infinite Series

Lesson Goal: Find the sum, if it exists, of an infinite geometric series or a telescoping series.

13. Write out the partial sums for the series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$ and use the partial sums to determine the convergence and the sum, if it exists..

14. Write out the partial sums of the series $\sum \frac{1}{4}$ and use the partial sums to determine the convergence and the sum, if it exists.

Geometric Series: Write the repeating decimal as a convergent geometric series and use the series to find its value as a fraction.

15. $0.0\overline{27}$

16. $0.24\overline{5} =$

Telescoping Series

17. Consider the series: $\sum \frac{4}{n(n+1)}$. Does it converge?

18. Misconception #1: $8+1-1+1-1+1-1...$ is a telescoping series.

19. Misconception #2: The sum of a convergent telescoping series is simply the sum of the terms that do not add out. NOT!

Example: $\left(\frac{1}{2}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{7}{8}\right)+\left(\frac{7}{8}-\frac{15}{16}\right)+\left(\frac{15}{16}-\frac{31}{32}\right)+\left(\frac{31}{32}-\frac{63}{64}\right)+...$

Determine if the series diverges or converges. If it converges, find the sum.

20. $\sum\left(\frac{2}{3}\right)^n$

21. $\sum\frac{2}{(n+1)(n+2)}$

Lesson Goal: Determine divergence of an infinite series using the Nth Term Test and determine the convergence or divergence of a *p*-series.

Can you make a determination of convergence or divergence on the following series?

22. $\sum\frac{3n}{500n+2}$.

23. $\sum\frac{3}{500n+2}$

***P*-Series: Determine the divergence or convergence of the series. Justify your answer. If possible, find the sum.**

24. $\sum_{1}^{\infty}\frac{8}{n^2}$.

25. $\sum\left(\frac{4}{3}\right)^n$

26. $\sum\frac{n-1}{2n+5}$

27. $\sum\frac{5}{(n-2)(n-3)}$

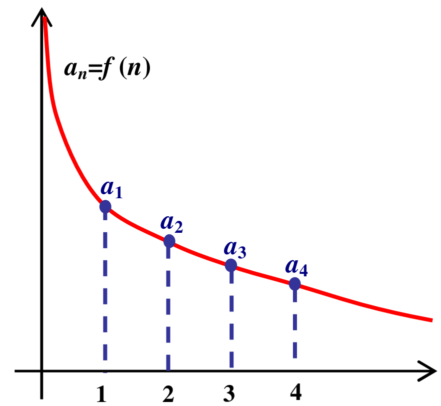
28. $\sum_{1}^{\infty}\frac{1}{\sqrt[3]{n}}$

Integral Test

Lesson Goal: Use the Integral Test to determine the convergence or divergence of an infinite series.

Thm. **Integral Test**: If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{1}^{\infty} a_n \text{ and } \int_{1}^{\infty} f(x)dx \text{ either both converge or both diverge.}$$



Proof: Consider the interval from 1 to n divided unto 1 unit subdivisions. Note that there would be $n-1$ subintervals. Now, since $f(x)$ is positive, continuous, and decreasing, then

$$\sum_{1}^{n} \text{ right rectangles} \leq \int_{1}^{n} f(x)dx \leq \sum_{1}^{n} \text{ left rectangles}$$

Each rectangle corresponds to a term of the sequence a_n. Hence $\sum_{2}^{n} a_n \leq \int_{1}^{n} f(x)dx \leq \sum_{1}^{n-1} a_n$

which can be rewritten as $S_n - a_1 \leq \int_{1}^{n} f(x)dx \leq S_{n-1}$

Case 1: If $\int_{1}^{\infty} f(x)dx = L$, then $S_n - a_1 \leq L$ or $S_n \leq L + a_1$. This guarantees that $\{S_n\}$ is bounded. Since we already know that $\{S_n\}$ is monotonic, then $\{S_n\}$ converges.

Case 2: If $\int_{1}^{\infty} f(x)dx = \infty$, then the inequality $\int_{1}^{n} f(x)dx \leq S_{n-1}$ implies that $\{S_n\}$ is unbounded. This guarantees that $\{S_n\}$ diverges.

NOTE: Since convergence or divergence of a series is not affected by deleting any finite number of terms, if the conditions for the Integral Test are not satisfied by all $n \geq 1$, but are satisfied by all $n \geq m$, where $m > 1$, then you can simply use the integral $\int_{m}^{\infty} f(x)dx$ to test for convergence or divergence.

CAUTION The value of the convergent integral is NOT the value of the convergent series.

29. Use the Integral Test to determine the convergence or divergence of the series $\sum_{1}^{\infty} \frac{2x}{x^2+3}$.

30. Use the Integral Test to determine the convergence or divergence of the series $\sum_{1}^{\infty} \frac{1}{x^2}$.

Determine the divergence or convergence of the series. Justify your answer.

31. $\sum_{1}^{\infty} \frac{2}{n^3}$

32. $\sum_{1}^{\infty} \frac{n-3}{-n+1}$

33. $\sum_{1}^{\infty} \frac{5}{4^n}$

34. $\sum_{1}^{\infty} \frac{\ln n}{n}$

35. $\sum_{1}^{\infty} \frac{100}{\sqrt[4]{n}}$

36. $\sum_{n=3}^{\infty} \left(\frac{3}{n-1} + \frac{3}{n-2} \right)$

37. $\sum \frac{4}{n^2+4n+3}$

Direct Comparison Test

Lesson Goal: Use the Direct Comparison Test or the Limit Comparison Test to determine the convergence or divergence of an infinite series.

38. Determine the convergence or divergence of the series $\sum_{1}^{\infty} \frac{1}{x^2+3}$.

39. Determine the convergence or divergence of the series $\sum_{1}^{\infty} \frac{6n^2+1}{2n^3-1}$.

40. Determine the convergence or divergence of the series $\sum_{1}^{\infty} \frac{1}{2n^2-n}$.

Limit Comparison Test

Compare the order of magnitude of each pair of series.

41. $\sum 4n^4+3n$ $\quad\quad$ $\sum 3n^2-9$

42. $\sum \frac{5}{n^5-7n^2}$ $\quad\quad$ $\sum \frac{8}{n^4+9n^3}$

43. $\sum \frac{6n}{n^2-9}$ $\quad\quad$ $\sum \frac{8}{n+2}$

44. $\sum n!$ $\quad\quad$ $\sum 8^n$

45. $\sum \frac{1}{2^{n!}}$ $\quad\quad$ $\sum \frac{1}{e^n}$

46. $\sum |\cos(n)|$ $\quad\quad$ $\sum n$

Use the Limit Comparison Test to determine the convergence or divergence of the series

47. $\sum_{1}^{\infty} \frac{1}{2n^2 - n}$.

48. $\sum_{1}^{\infty} \frac{n^2 - 3}{n^3 + n^2}$

Determine the divergence or convergence of the series.

49. $\sum_{1}^{\infty} \frac{n}{n^4 + 3}$

50. $\sum_{1}^{\infty} \sin\left(\frac{1}{n}\right)$

51. $\sum_{1}^{\infty} \frac{\ln(n)}{n}$

52. $\sum_{1}^{\infty} \frac{\ln(n)}{n^2}$

Alternating Series

Lesson Goal: Determine if an alternating series is convergent or divergent.

53. Show that the alternating harmonic series is convergent.

54. It is important to recognize that both conditions in the Alternating Series Test must be met. How does each series below fail to meet the conditions?

a. $\frac{3}{1}-\frac{4}{2}+\frac{5}{3}-\frac{6}{4}+\frac{7}{5}-\frac{8}{6}+\ldots.$

b. $\frac{3}{1}-\frac{2}{1}+\frac{3}{2}-\frac{2}{2}+\frac{3}{3}-\frac{2}{3}+\frac{3}{4}-\frac{2}{4}+\frac{3}{5}-\frac{2}{5}\ldots$

55. Use the Alternating Series Test to show that $\sum_{1}^{\infty}\frac{(-1)^{n-1}}{e^n}$ converges.

Which of the following is an alternating series?

56. $\sum_{1}^{\infty}(-1)^n\left(-\frac{1}{5}\right)^{n+1}$

57. $\sum_{1}^{\infty}\cos(n\pi)$

58. $\sum_{1}^{\infty}(-1)^{n+1}\cos(n)$

How many terms would you have to sum in order to have a value that is within 0.01 of the true sum?

59. $\sum_{1}^{\infty}\left(\frac{-2}{3}\right)^{n-1}$

Determine the divergence or convergence of the series. If it is a convergent alternating series, determine the number of terms that would have to be summed to be accurate to the nearest tenth.

60. $\sum_{2}^{\infty}\frac{(-1)^{n+1}n!}{\ln(n)}$

61. $\displaystyle\sum_{1}^{\infty}\frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$

62. $\displaystyle\sum_{1}^{\infty}\frac{1-\cos(n)}{n^2}$

63. $\displaystyle\sum_{1}^{\infty}\frac{(-1)^n 2^n}{n!}$

Absolute and Conditional Convergence

Lesson Goal: Classify a series as absolutely convergent or conditionally convergent.

- The alternating harmonic series would be an example of a conditionally convergent series.
- The series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+...$ would be an example of an absolutely convergent series.

Determine the divergence or convergence (conditional or absolute) of the series. If it is a conditionally or absolutely convergent alternating series, find the number of terms that would have to be summed to have a sum accurate to the nearest thousandth.

64. $\displaystyle\sum_{3}^{\infty}\frac{(-1)^{n+1}}{(n-2)^{1/4}}$

65. $\displaystyle\sum_{1}^{\infty}\frac{\sin(n)}{n^3}$

66. $\displaystyle\sum_{2}^{\infty}\frac{(-1)^{n+1}e^{n+1}}{2^n}$.

67. $\displaystyle\sum_{1}^{\infty}\frac{(-1)^n\sqrt{n}}{(n+1)!}$

68. AP Multiple Choice: Which of the following series are convergent?

I. $1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+...$

II. $1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$

III. $1-\frac{1}{3}+\frac{1}{3^2}-...+\frac{(-1)^{n+1}}{3^{n-1}}+...$

a. I only b. III only c. I and III only d. II and III only e. I, II, and III

69. AP Multiple Choice: Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{8}{n^{1.5}}$ II. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+100}}$ III. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

a. I only b. III only c. I and II only d. I and III only e. I, II, and III

Ratio Test

Lesson Goal: Use the Ratio Test to determine the convergence or divergence of an infinite series.

Use the Ratio Test to determine if the series converges or diverges.

70. $\sum_{1}^{\infty}\frac{1}{n!}$

71. $\sum_{1}^{\infty}\frac{e^n}{n^5}$

72. $\sum_{1}^{\infty}\frac{n^2 3^n}{n!}$

73. $\sum_{1}^{\infty}\frac{n!}{n^2}$

Determine the converge or divergence of each.

74. $\sum_{1}^{\infty}\frac{2^{n+1}}{3^n}$

75. $\sum_{1}^{\infty}\frac{n!}{n^n}$

76. $\sum_{1}^{\infty} \frac{\sqrt{n}}{(n+1)^2}$

77. $\sum_{1}^{\infty} \frac{n^e}{e^n}$

78. $\sum_{1}^{\infty} \frac{\text{arccot } n}{n}$

79. $\sum_{1}^{\infty} \frac{1}{n \ln n}$

80. $\sum_{3}^{\infty} \frac{(-1)^n n^2}{(n-2)(n+5)}$

Polynomial Approximations for Transcendental Functions

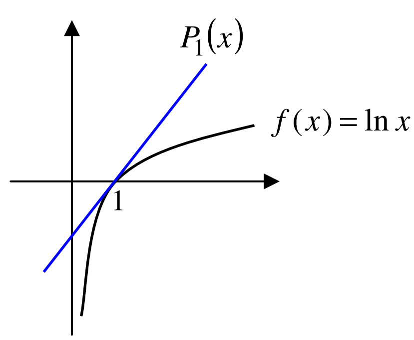
How does a calculator or computer compute all those values for trigonometric and logarithmic functions??? Much of the work is done with polynomial approximations to these functions.

Taylor Polynomial Approximations

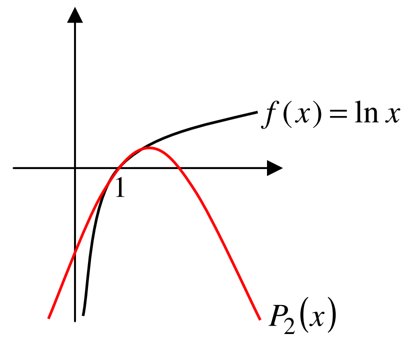
Lesson Goal: Find the Taylor Polynomial approximation of an elementary function.

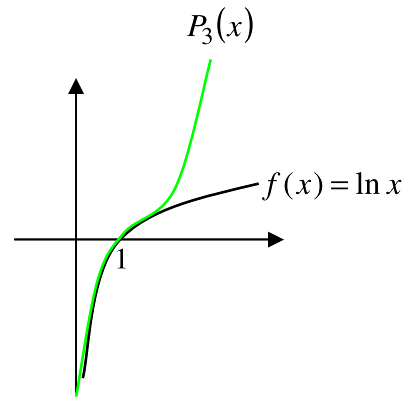
81. Suppose we want to approximate $f(x) = \ln x$ by a polynomial. We cannot center it at zero because the function does not exist at zero. So we will choose $x = 1$ instead, and write a polynomial of degree 3.

a. Start with a linear approximation. We know it as the tangent line. It is an approximation that has two requirements: the line must pass through the given point and the line must have the same slope as the slope of the curve. Another way of saying that is that the linear approximation (or tangent line) is a polynomial $P_1(x)$ where $P_1(1) = f(1)$ and $P_1'(1) = f'(1)$.



b. Now suppose that we want a better approximation. Let's try a quadratic approximation using the same structure $P_2(x) = a_0 + a_1(x-1) + a_2(x-1)^2$ with the requirement that $P_2(1) = f(1)$, $P_2'(1) = f'(1)$, and $P_2''(1) = f''(1)$.





c. Now try a third degree polynomial $P_3(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$ where $P_3(1) = f(1)$, $P_3'(1) = f'(1)$, $P_3''(1) = f''(1)$, and $P_3'''(1) = f'''(1)$.

d. The pattern that we need to see is

82. Find the 7th degree Taylor polynomial for $f(x) = \sin x$ centered at $x = 0$. Use it to find an approximation for $\sin\left(\frac{\pi}{3}\right)$.

Lesson Goal: Find the Taylor Polynomial approximation of an elementary function.

83. Find the n^{th} degree Taylor polynomial for $y = \dfrac{1}{1-x}$ centered at $x = 2$.

Lesson Goal: Find the error in computing a functional value using the n*th* Taylor polynomial expansion of a function.

84. Given a function $f(x)$ whose fourth-degree Taylor polynomial expansion at $x = 0$ is $P(x) = 2 - \frac{3}{4}x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{16}x^4$.

   a. Does $f(x)$ have a critical point at $x = 0$?

   b. Does $f(x)$ have a critical point at $x = 1$?

   c. Is $f(x)$ concave upward or downward at $x = 0$?

85. Use TI-CAS to find the 5^{th} degree Taylor polynomial for $y = \ln(x+1)$ centered at $x = 0$.

   (*menu*) *Calculus* *Series* *Taylor Polynomial* ln(x + 1) , x , 5 , 0) (*enter*)

86. Use TI-CAS to evaluate the polynomial at $x = 1.5$.

   taylor(ln(x + 1), x, 5, 0) | $x = 2$ (*enter.*)

Lagrange Error Bound

87. Give a bound on the error when $\ln(x+1)$ is approximated by its fifth-degree Taylor polynomial about 0 for $x = 2$.

88. Try it again for $x = 0.5$.

89. How many terms would be required in the Taylor approximation of $\sin x$ centered at $x = 0$ to approximate $\sin 2°$ accurate to four decimal places?

90. Find those two terms and use them to approximate $\sin 2°$.

91. Give a bound on the error when e^x is approximated by its fourth-degree Taylor polynomial about 0 for $-0.5 \le x \le 0.5$.

Lesson Goal: Find the radius of convergence of a power series.

Power Series Examples

- $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ is a power series centered at 3.
- $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ is a power series centered at 0. (Remember that $0!=1$ by definition)
- $\sum_{n=0}^{\infty} \frac{1}{x!}(x+1)^n$ is **NOT** a power series.

92. Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}(x-3)^n$. Then:

a. $f(1) =$

b. $f(2) =$

c. $f(3) =$

d. $f(3.5) =$

e. $f(4) =$

Note that in expanding a series it is often best to write the terms without reducing so that the pattern is emphasized.

93. Which of the above series appear to converge?

Find the radius of convergence for each.

94. $f(x)=\sum_{n=0}^{\infty}\frac{1}{n+1}(x-3)^n$

95. $f(x)=\sum_{n=0}^{\infty}2^{2n}x^{2n}$

96. $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$

97. $f(x)=\sum_{n=0}^{\infty}\frac{n!}{2^n}(x+2)^n$

Find the interval of convergence for each.

Lesson Goal: Determine the endpoint convergence of a power series and use it to state the interval of convergence.

98. $f(x)=\sum_{n=0}^{\infty}\frac{1}{n+1}(x-3)^n$ In the previous lesson we found the radius of convergence to be 1.

99. $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{\ln n}$

100. $f(x)=\sum_{n=0}^{\infty}\frac{n}{2^n}(x+1)^n$

101. If $f(x)=1+x+x^2+x^3+...+x^n+...$, find:

a. Radius of convergence.

b. $f'(x)$

c. $\int f(x)dx$

102. If $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$, find:

a. Radius of convergence.

b. $f'(x)$

c. $\int f(x)dx$

d. What interesting properties does this series have?

e. What traditional function do you know with these properties?

Geometric Power Series

Lesson Goal: Create a new power series from a given power series, using substitution, algebraic operations, integration, or differentiation.

103. Write a power series expansion centered at $x = 0$ for $f(x) = \dfrac{5}{1-x}$.

104. Write a power series expansion centered at $x = 0$ for $f(x) = \dfrac{3x}{1-x^2}$.

105. Since $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$, write a power series expansion for $\arctan x$.

106. If $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}(x-3)^n$,

a. Find the interval of convergence.

b. Write out the first three terms and the general term of the power series expansion for $g(x) = \dfrac{f(x)-1}{x-3}$.

c. Write out the first three terms and the general term of the power series expansion for $h(x) = f'(3x)$.

107. Recall the earlier expansion, centered at $x=0$ that we wrote for $f(x)=\dfrac{5}{1-x}=5+5x+5x^2+5x^3+...$ Now create another power series for this same function centered at $x=2$ and find its interval of convergence.

Taylor and Maclaurin Series

Lesson Goal: Create a Taylor or Maclaurin series using another Taylor series, by methods of substitution, differentiation, and/or integration.

108. Show that the Maclaurin series for $\cos x$ converges for all real numbers.

Power Series for Elementary Functions

The AP requires you to memorize these four Taylor Series with their intervals of convergence.

- $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+...+\dfrac{x^n}{n!}+...$ where $I=(-\infty,\infty)$.
- $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+...+\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}+...$ where $I=(-\infty,\infty)$.
- $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+...+\dfrac{(-1)^n x^{2n}}{(2n)!}+...$ where $I=(-\infty,\infty)$.
- $\dfrac{1}{1-x}=1+x+x^2+x^3+...+x^n+...$ where $I=(-1,1)$.

109. Find a Taylor series for $\sin(3x)$ centered at $x=0$.

110. Find a Maclaurin series for $\int \sin x^2 dx$.

111. Find a Maclaurin series for $x\cos\left(x^3\right)$.

112. Find a Taylor series for $\cos(2x+\pi)$ centered at $x=0$.

113. Evaluate: $1+3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+...$

114. Evaluate: $1-0.1+0.1^2-0.1^3+...$

115. Evaluate: $1-\frac{100}{2!}+\frac{10000}{4!}-\frac{1000000}{6!}+...$