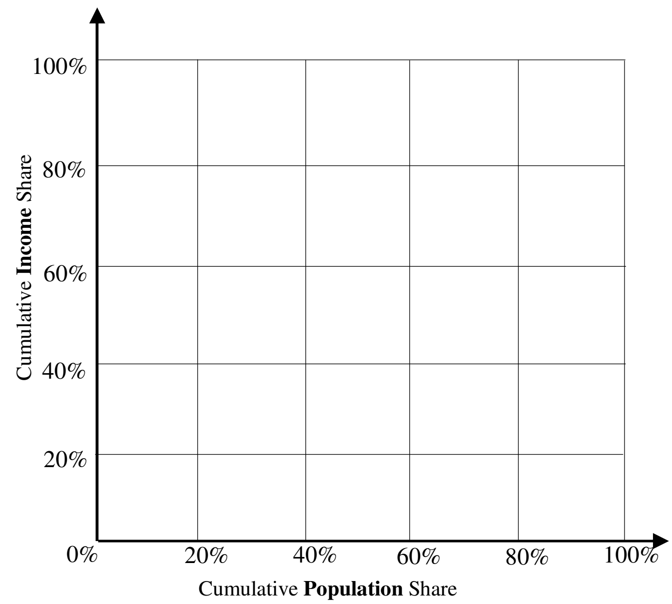
CLASSWORK 7 – Applications of Integration

Gini Index

Lesson Goal: Calculate a Gini Index and understand its value in analyzing economic trends.

1. Consider a population where every worker earns the same income.

   a. Make a graph of cumulative population share versus cumulative income share, distributing the population into quintiles. List the ordered pairs.



   b. Note that the graph is a simple diagonal line. It is called the *Line of Income Equality* or the *Egalitarian Line*.

2. Now consider these statistics for the U.S. population in 1988. Assume that incomes have been ordered from least to greatest.

   lowest quintile of the population had 4.6% of the income.

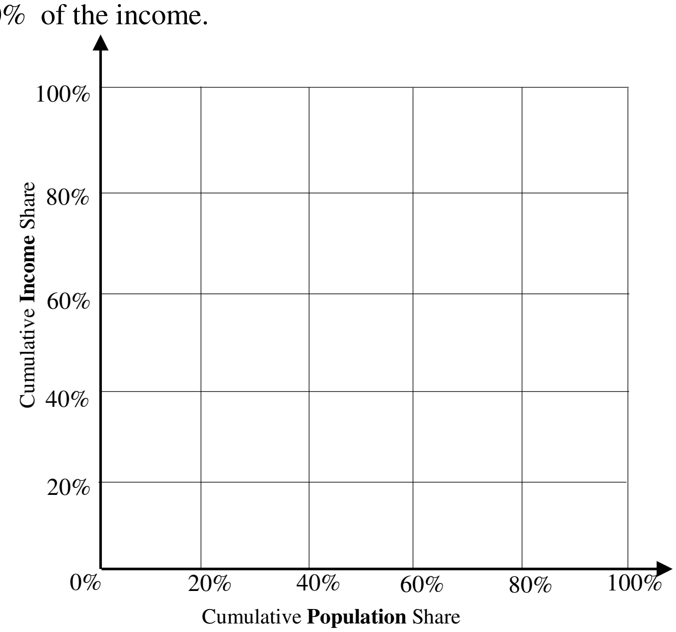
   2nd quintile of the population had 10.7% of the income.

   3rd quintile of the population had 16.7% of the income.

   4th quintile of the population had 24.0% of the income.

   highest quintile of the population had 44.0% of the income.

   a. Use these values to plot cumulative population share versus cumulative income share. List the ordered pairs.



   b. Find a best-fit curve for the data.

   c. This curve is called the *Lorenz Curve*.

d. Calculate the area between the Lorenz curve and the Egalitarian Line

e. The Gini Index is a relative index of income inequity. $0 \le \text{Gini Index} \le 1$. The larger the value of the Gini Index, the more inequitable the income distribution.

3. Lab Demonstration: Perform the calculations on a StudyWorks worksheet with the data from 1929.

   lowest quintile of the population had 0.03% of the income.

   2nd quintile of the population had 12.47% of the income.

   3rd quintile of the population had 13.8% of the income.

   4th quintile of the population had 19.3% of the income.

   highest quintile of the population had 54.4% of the income.

Building an Integral

Lesson Goal: Build an integral to solve a problem.

4. Suppose that someone has done a population study in Pennsylvania and they have developed some formulas describing the population density in and around Newtown Square. We are interested in the formula $D(x)$ that describes the population density along the 20-mile stretch of Route 3 that extends from Philadelphia west to West Chester. For example, the value $D(6)$ tells how many people there are per square mile if you are 6 miles west of Philadelphia. We are going to use this formula to calculate the total population along a 2 mile wide path following Route 3 from Philadelphia to West Chester.

5. Suppose someone has a well-used dartboard and they have studied the distribution of holes on the dartboard. Their findings are that the holes are clustered much more closely at the center of the board and more sparsely towards the edges. Indeed the density of holes can be described as a function $D(r)$ where r is the distance in centimeters from the center of the board. The board has a radius of 40 cm. We want to calculate the total number of holes on the board.

Volumes of Solids of Known Cross-Section

Lesson Goal: Find the volume of a solid of known cross section.

6. The base of a solid is a circle of radius 1. Parallel cross sections cut perpendicular to the base form rectangles whose height is 2. Find the volume of the solid.

7. The base of a solid is an equilateral triangle of side 2. Parallel cross sections cut perpendicular to the base of the triangle form a square. Find the volume of the solid.

8. The base of a solid is a circle of radius 2. Parallel cross sections cut perpendicular to the base are equilateral triangles. Find the volume of the solid.

9. The base of a solid is a region bounded by the curve $y = 4 - x^2$ and the x-axis. Parallel cross sections cut perpendicular to the base and perpendicular to the x-axis are semi-ellipses of height 6. Find the volume.

Volumes of Solids of Revolution – Disc Method

Lesson Goal: Find the volume of a solid of revolution using the disc method.

10. Given a region bounded by the curve $y = x^2$, $y = 0$, and $x = 2$. Find the volume of the solid formed by revolving the region about the *x*-axis.

11. Given a region bounded by the curve $y = x^3$, $y = 1$, and $x = 0$. Find the volume of the solid formed by revolving the region about the *y*-axis.

12. Given a region in the *first* quadrant bounded by the curve $y = x^2$ and $y = \sqrt{x}$. Find the volume of the solid formed by revolving the region about the *x*-axis.

13. Find the volume of the solid generated by revolving the region bounded by $y = e^x$, the *y*-axis, and $y = e$ about the *x*-axis.

Lesson Goal: Find the volume of a solid revolved about a line other than an axis.

14. Given a region bounded by the curve $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid formed by revolving the region about the line $x = 1$.

15. Given a region bounded by the curve $y = x(x-2)^2$ and $y = 0$. Find the volume of the solid formed by revolving the region about the line $y = 2$.

16. Given a region in the second quadrant bounded by the curve $x = y^3 - y$ and $x = 0$. Find the volume of the solid formed by revolving the region about the line $x = -2$.

17. Given a region in the *first* quadrant bounded by the curve $y = x^2 + 3$, $y = 0$, $x = 0$, and $x = 2$. Find the volume of the solid formed by revolving the region about the *y*-axis.

Arc Length

Lesson Goal: Find the length of an arc.

18. What is meant by the length of an arc?

19. How would we approximate the length of an arc?

20. Suppose we subdivide an arc into n pieces. Let Δs_i represent one subdivision of the arc length.

    Then $\Delta s_i \approx \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ by ______________________________

    Adding them all up gives an approximation for the arc length s:

    $$s \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

    Now rewrite the radical expression by factoring out $(\Delta x_i)^2$:

    $$s \approx \sum_{i=1}^{n} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

    By the Mean Value Theorem there exists a number $c \in (x_{i-1}, x_i)$ such that $f'(c_i) = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$ or

    $f'(c_i) = \dfrac{\Delta y_i}{\Delta x_i}$.

    Substituting this into the summation we get:

    $$s \approx \sum_{i=1}^{n} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

21. To get an exact value, we take the limit of the summation as the norm of the partition approaches zero.

    Thus $s = \lim\limits_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i)$

    or $s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

22. Find the length of the arc of

$y = 2(x-1)^{\frac{3}{2}} - 4$ from $x = 2$ to $x = 5$.

23. Find the length of the arc of

$x = \frac{1}{3}(y-3)\sqrt{y}$ from $0 \le y \le 3$.

24. A particle is traveling around a coordinate system in a continuous loop. The loop is formed by the parts of the two curves $y = \ln x$ and $y = (x-3)^2$ that fall between their points of intersection. Find the distance that the particle travels on one loop around.

Work Done to Lift a Liquid

Lesson Goal: Use an integral to solve a problem involving work to move a liquid vertically.

25. If a 50 pound weight is lifted 6 feet, find the work done.

26. A tank having the shape of a right circular cylinder of altitude 8 feet and radius of base 5 feet is full of water. Find the work done to pump the water over the top edge.

27. An inverted right circular cone with radius 3 feet and height 10 feet is full of water. Find the work required to pump all of the water out of the top of the tank and into a second tank whose top is 5 feet above the top of the cone.

28. A trough 6 feet long is half full of water. Its cross section is in the shape of a semi-circle with diameter 2 ft at the top. If the trough develops a leak at the bottom, how much work is done as the trough empties?

Work Done Lifting a Chain

Lesson Goal: Use an integral to solve a problem involving work to lift a handing object.

29. A 20 foot chain weighing 1/2 lb/ft hangs from the top of a tall building. How much work is done in pulling the chain to the roof of the building?

30. A 50 foot chain weighing 2 lb/ft is hanging vertically. How much work is done in lifting one end of the chain to the level of the other end?

31. A steam shovel is excavating sand. Each load weighs 500 lb when excavated. The shovel lifts each load to a height of 15 feet in $\frac{1}{2}$ min, then dumps it. A leak in the shovel lets sand drop out while the shovel is being raised. The rate at which the sand leaks out is 160 lb/min. Find the amount of work done in raising one load.

32. A 20 foot chain weighing 1/2 lb/ft hangs from the top of a 10 foot building (with 10 feet of chain on the ground) How much work is done in pulling the chain to the roof of the building?

Fluid Force

Lesson Goal: Use an integral to find the fluid force on a vertical surface.

33. Find the force on the bottom of a swimming pool $40 \times 20 \times 6$ feet deep.

34. Now consider the force on the end wall of the swimming pool.

35. The face of a dam adjacent to the water has the shape of an isosceles trapezoid of altitude 20 feet, upper base 50 feet and lower base 40 feet. Find the total force exerted by the water on the dam when the water is 15 feet deep.

36. The vertical ends of a trough are in the shapes of semicircles of diameter 2 feet with diameter horizontal. Find the total force on one end when the trough is full of water.

37. Find the fluid force on a vertical side of a tank if the tank is full of water and the side has the shape of a right triangle whose legs are 5 feet and 10 feet. The longer leg is positioned horizontally at the bottom of the tank.