CLASSWORK 6 – Differential Equations

Lesson Goal: Draw a slope field and use it to visualize the solution to a differential equation.

1. Solve the differential equation: $\frac{dy}{dx} = 3x^2 + 4x$.

2. Construct a slope field for the differential equation: $\frac{dy}{dx} = 2x$ and draw the specific solution where $f(-3) = 2$

   a. Pick any point on a coordinate grid and use the coordinates of the point to calculate the derivative at that point.

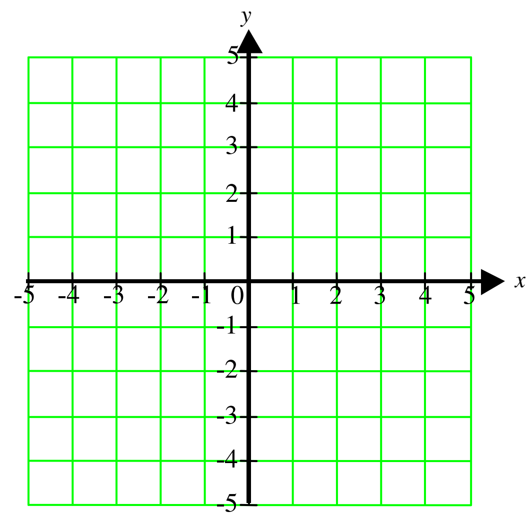
   b. Draw a short tangent line segment on the coordinate system at the chosen point.

   c. Repeat for multiple points on the coordinate system.

   d. Try to visualize the function from the slope field. Notice that many curves can be drawn depending on where one starts. The starting point is what we call the initial condition.

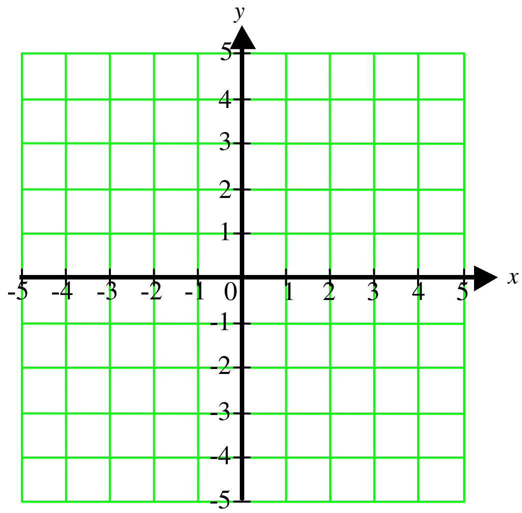
   e. Draw a specific solution where $f(-3) = 2$.

x	y	dy/dx



3. Construct a slope field for the differential equation: $\frac{dy}{dx} = \frac{1}{x}$ and draw several solutions.

x	*y*	*dy/dx*



4. Use the TI-CAS to draw a slope field for $\frac{dy}{dx} = \frac{1}{x}$ and sketch a specific solution with $f(0.1) = -3$.

a. Sketch the slope field:

In the graphing window: (*menu*) *Action* *Delete All* (*enter*)

(*menu*) *Graph Type* *Diff Eq* (*enter*) (*esc*)

Zoom Standard

In the entry line beside $y1' =$ type $1/x$ (*enter.*) Make sure that there is nothing typed into the parentheses beside the symbol $(x_0, y1_0)$.

In the entry line (*tab*) until you get to symbol [...] (*enter*)

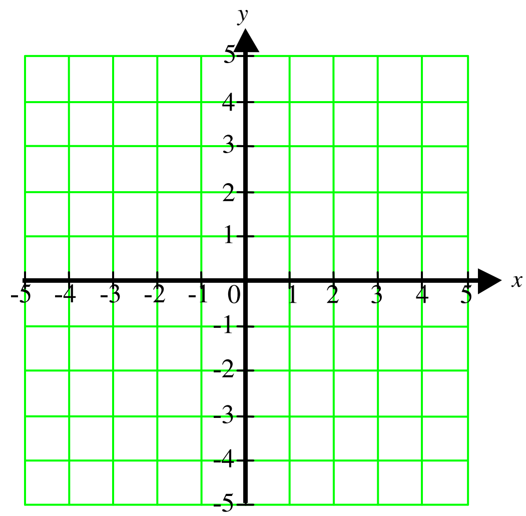
Scroll down to the bottom and beside *Field Resolution* type *20* *OK*

b. Draw a specific solution for $f(0.1) = -3$.

In the entry line beside $(x_0, y1_0)$ type $0.1, -3$ (*enter*)

5. Construct a slope field for the differential equation: $\frac{dy}{dx} = 2 + y$ and sketch the specific solution where $f(-2) = -1.5$.

x	*y*	*dy/dx*



6. Verify the previous result by using the TI-CAS to draw a slope field for $\frac{dy}{dx} = 2 + y$ and draw a specific solution with $f(-2) = 1.5$.

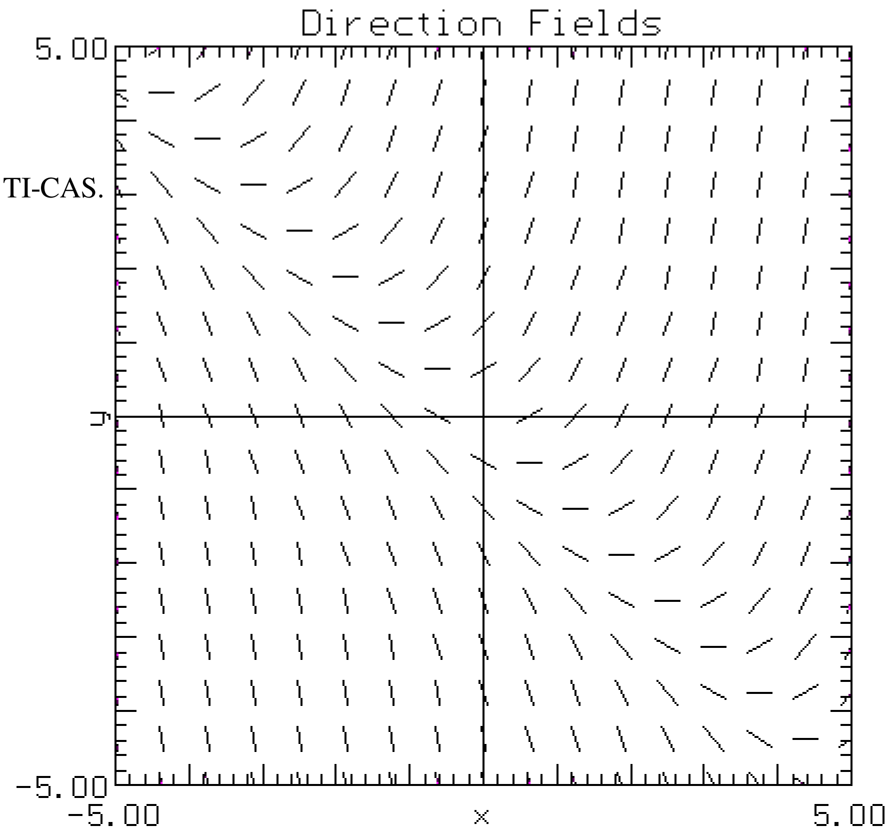
   In the entry line beside $y1' =$ type $2 + y1$ Beside $(x_0, y1_0)$ type $-2, 1.5$ (*enter.*)

7. Use the slope field below to draw a solution to the differential equation $\frac{dy}{dx} = x + y$

   a. Where $f(-2) = 1.5$.

   b. Where $f(3) = 0$.

   c. Check both solutions on the TI-CAS.



Euler's Method

Lesson Goal: Use Euler's Method to find a numerical solution to a differential equation.

8. Many populations grow at a rate which is directly proportional to the population itself. Assume this pattern of growth for the world population. In 2005, the constant of proportionality was 0.013, with a world population estimate of 6.396 billion. What would you project the population to be in the year 2020?

   a. Write an expression for the derivative.

   b. Write an expression for the differential.

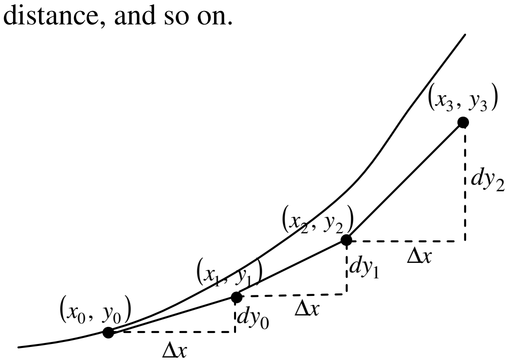
   c. Compute dy over a fifteen year interval.

   d. Why would we expect this value to be significantly off?

e. Compute dy over a *one* year interval, from 2005 to 2006.

f. Compute dy over a one year interval, from 2006 to 2007.

In the previous lesson we saw how to sketch a solution curve to a differential equation using its slopefield (or direction field), whose line segments are everywhere tangent to the solution. Now we will do the same thing numerically, computing points on the solution curves. This process is called Euler's Method. Think of the direction field as a set of signposts directing us across the plane. Pick a starting point (corresponding to the initial value), and calculate the slope at that point using the differential equation. This slope is a signpost telling us the direction to take. Head off along a straight line path for a short distance in that direction. Stop and look at the new signpost. Recalculate the slope from the differential equation, using the coordinates of the new point. Change direction to correspond to the new slope, and move another small distance, and so on.



This can be summarized as:

(new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).

Let $y'(x_i)$ be the slope of the function at the point (x_i, y_i). Then Euler's Formula is $y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$. Remember that these values are always approximations to the actual curve. The smaller the value of Δx, the better the approximation.

g. Use TI-CAS to calculate the recursive pattern.

In the calculator window type 6.396 (enter)

(ctrl)(–) + 0.013• (ctrl)(–) •1 (enter)

(enter)

(enter)...

Continue in this manner until you get a value for the year 2020. (Ans: 7.763 billion)

h. Redo with a 6-month time interval.

9. Interest that is compounded continuously yields a differential equation of the form $\frac{dy}{dt} = ky$ where k is the rate of interest and y is the amount in the account. Suppose you have \$1000 invested at 6% interest compounded continuously.

   a. Use the formula $A = Pe^{rt}$ to verify that $\frac{dA}{dt} = rA$.

   b. Using Euler's Method with a 6-month interval of time, write out the calculation for finding y_1.

   c. Using Euler's Method with 6-month intervals of time, write a general formula for y_i.

   d. Using your TI-CAS and Euler's Method with 3-month time intervals, find the total amount of money you would have in 4 years.

10. A sky diver jumps from an airplane. During the free-fall stage, her speed increases at the acceleration of gravity, about 32.16 ft/sec^2. But wind resistance causes a force that reduces the acceleration. The resistance force is proportional to the square of the velocity. Assume that the constant of proportionality is 0.0015, so that $\frac{dv}{dt} = 32.16 - 0.015v^2$, where v is in feet per second and t is in seconds.

    a. If the diver's initial velocity was zero, find her velocity in 10 seconds, using Euler's Method with half-second intervals of time.

    b. If you changed to quarter-second intervals of time, would the value calculated for her velocity in 10 seconds be larger or smaller than that calculated using half-second intervals of time? Verify your hunch by doing the calculation.

Lesson Goal: Solve a simple differential equation analytically using separation of variables.

11. Solve the differential equation: $\frac{dy}{dx} = 3x^2 + 4x$.

12. Solve the differential equation $\frac{dy}{dx} = 3x^2 - 6x + 9$ for the specific solution where $f(3) = 5$.

13. Find a function $f(x)$ for which $f''(x) = 6$, $f'(1) = \frac{27}{4}$, and $f(4) = 60$.

14. Find the particular solution of $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ that contains the ordered pair (1, 2).

15. Find the particular solution of $\frac{dy}{dx} = \pi$ that contains the ordered pair (1, 1).

16. Find the particular solution of $\frac{dy}{dx} = \frac{1}{x-2}$ that contains the ordered pair (3, 3). State the domain of the solution.

17. Find the general solution of $\frac{dy}{dx} = y$.

18. Find the particular solution of $y' = \frac{x}{y} - \frac{1}{y}$ where $y(0) = -1$.

19. Find the general solution of $\frac{dy}{dx} = 6 + 4y$.

20. Find the general solution of $y' = \frac{\ln x^y}{x}$.

21. Find the general solution of $y' = ky\left(1 - \frac{y}{1000}\right)$ using your TI-CAS.

   (menu) *Calculus* *Differential Equation Solver* y $'$ $= k \times y \times (1 - y \div 1000\ ),\ x\ ,\ y\ )$ (enter)

   Touchpad: Derivative symbol is a menu choice on [?!]
   Clickpad: Derivative is a grey post on the upper right part of the keypad.
   (Don't use implied multiplications; they will cause problems.)

22. Find the particular solution of $\frac{dy}{dt} = \frac{3t}{y}$ with $y(2) = 5$, using your TI-CAS.

   (menu) *Calculus* *Differential Equation Solver* $y' = 3 \times t \div y$ ⎵ *and* ⎵ $y\ (\ 2\ ) = 5\ ,\ t\ ,\ y\ )$ (enter)

Exponential Growth and Decay

Lesson Goal: Solve a verbal problem involving exponential growth or decay.

23. In a favorable environment, the number of bacteria increases at a rate proportional to the number present. If 1000 bacteria are present at a certain time and 2000 are present an hour later, find the number present four hours later.

24. A bank account grows continuously at a rate of 4.5%. If \$3000 is deposited in the account, how much will be in the account 6 years later? What is the effective annual interest rate?

25. A radioactive substance decays at a rate proportional to the amount present. If one gram of radioactive substance reduces to $\frac{1}{4}$ gm. in four hours, find:

    a. The half-life of the substance.

    b. How long it will take until only $\frac{1}{10}$ gm. remains.

Learning Curves and Cooling Curves

Lesson Goal: State the properties of a learning curve or cooling curve.

26. Use TI-CAS to draw a slope field for the differential equation $\frac{dy}{dt} = -k(y - E)$ where $k = 1$ and $E = 3$.

27. How would you find the equilibrium solution?

28. Let y be the temperature of a body that is cooling and T be the temperature of the surrounding air. Does the differential equation $\frac{\frac{dy}{dt}}{y - T} = k$ fit the form of a stable equilibrium? Is this k negative? Why?

29. What is the equilibrium solution?

30. Does the differential equation $\frac{dQ}{dt} = \frac{-r}{V}Q$ fit the form of a stable equilibrium? Is this k negative? Why?

31. What is the equilibrium solution?

32. A body originally at $120°$ Fahrenheit cools to $100°$ in 10 minutes in air at $60°$. Find the temperature of the body after it has cooled for 1 hour.

33. Suppose a lake of volume 2700 km^3 has an outflow of 90 km^3/yr. How long will it take for 60% of its pollutants to be removed?

Lesson Goal: State the properties of a logistic curve and solve a logistic curve with Euler's Method.

34. Consider the differential equation: $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$ where k and L are both positive.

    a. Describe its pattern of growth.

    b. Generate a TI-CAS slope field, letting $k = 1.4$ and $L = 5$.

       In the graphing window: (*menu*) *Action* *Delete All* (*enter*)

       (*menu*) *Graph Type* *Diff Eq* (*enter*) (*esc*)

       Change the window settings to: $-8 \le x \le 8$ and $-5 \le y \le 8$ y-scale 1

       In the entry line beside $y1' =$ type $1.4 \times y1 \times (\ 1 - y1 \div 5\ )$ Make sure that there is nothing typed into the parentheses beside the symbol $(x_0, y1_0)$.

    c. Is this constrained or unconstrained growth? A stable or unstable equilibrium?

35. Where will a logistic curve have its fastest rate of growth?

36. Can you guess the y-coordinate of the inflection point?

37. A 10,000 acre forest has a carrying capacity of 1000 deer. Assumer that the deer population grows logistically with a growth constant of $k = 0.4$. Write a differential equation to model the population of deer in the forest.

38. Suppose a dorm has a total population of 500 students, and after a vacation 5 people return with the flu. The rate of spread of the flu is believed to be proportional to the product of the number of people who have had the flu and the number who haven't. Let F be the number of students who have had the flu at any time t (in days).

   a. Write a differential equation that describes the rate at which the flu spreads, assuming no one is immune.

   b. Is this a logistic pattern of growth? Give both a mathematical and a logical answer.

   c. Use Euler's Method to write a recursive expression for F.

   d. Suppose that by the time 20 students have had the flu, the rate of infection is about 9 students per day. Find the value of k.

   e. Using half-day intervals, use Euler's Method to determine the number of students who have had the flu by the 6^{th} day?

   f. How many days will it take until everyone has had the flu?

Solve using Partial Fractions

Lesson Goal: Use partial fractions to integrate a rational function.

39. $\int \frac{1}{1-x^2}dx =$

40. $\int \frac{x^3 - x^2 + 2x + 3}{x^2 + 3x + 2}dx =$

41. $\int \frac{\cos x dx}{\sin^2 x - 2\sin x - 3} =$

42. $\int \frac{dx}{(x-2)(x-3)(x+2)} =$

43. $\int_1^3 \frac{x^2 - x + 1}{x^2 + x}$

44. $\int \frac{e^t dt}{e^{2t} + 3e^t + 2} =$

45. $\int \frac{1}{x^3 - x^2} dx =$

46. $\int \frac{e^t dt}{e^{2t} + 2e^t + 1} =$

Lesson Goal: Solve a logistic differential equation analytically.

47. Suppose a dorm has a total population of 500 students, and after a vacation 5 people return with the flu. The rate of spread of the flu is believed to be proportional to the product of the number of people who have had the flu and the number who haven't. Let F be the number of students who have had the flu at any time t (in days).

    a. Write a differential equation that describes the rate at which the flu spreads, assuming no one is immune.

    b. Solve the differential equation using separation of variables and partial fractions.

    c. If after 3 days, 20 students have had the flu, how many days will it take until everyone has had the flu?

    d. On what day is the flu spreading the fastest?

48. Riverdale High has 1000 students. On day 0, Archie, Jughead, Betty, Veronica start a rumor, which spreads logistically. A day later, 25 students have heard the rumor.

    a. Why is it believable that this would be logistic growth?

b. Write the differential equation that describes the spread of the rumor. Let R be the number of students who have heard the rumor in time t (in days).

c. Solve the differential equation.

d. How many students will know 4 days later?

e. How many students will know 10 days later?

f. On what day is the rumor spreading the fastest?

Area Between Two Curves

Lesson Goal: Find the area between two curves.

49. Find the area between $y = -x(x+2)$ and $y = -5x(x+2)$.

50. Find the area between $y = 4x - x^2$ and $y = x$ from $x = 0$ to $x = 4$. Check answer with TI-CAS.

51. Find the line $x = h$ that divides the first quadrant region bounded by the two curves into two regions of equal area. $y = e^x - 1$ and $y = 2\sin x$.

52. Find the area between $y^2 = -x + 3$ and $y = -2x$. Evaluate integral with TI-CAS.

Lesson Goal: Find the area between two curves using horizontal rectangles.

53. Find the area of the region bounded by $x = y^2 - 7y + 6$ and the y-axis.

54. Find the area of the region bounded by $y^2 = -x + 3$ and $y = -2x$. Use TI-CAS to evaluate integral.

55. Find the area of the region bounded by $x = y^2 - 9$ and $x = \ln y + 2$.

56. Find the area of the region bounded by $y = x^3 + 2x^2 - 4x$ and $y = -x$.