CLASSWORK 4 - Integration

Approximating Area Under a Curve

Lesson Goal: Review methods of finding area under a curve.

1. Find the area (by hand) under the curve $y = e^x$ from $x = 1$ to $x = 3$. using 4 rectangles and evaluating the function at *left* endpoints.

2. Write out the setup to approximate the same area using *midpoint* approximations. Evaluate using TI-CAS AREAPROX program.

Determining Area Exactly

3. Find the area between the curve of $y = x^2 - 1$ and the *x*-axis from $x = -1$ to $x = 5$, using the limit of a summation.

Evaluating a Definite Integral

4. Write a single integral to represent the area between the curve of $y = x^2 - 1$ and the *x*-axis from $x = -1$ to $x = 5$. Evaluate the integral using the TI-CAS integral tool.

5. Evaluate the definite integral two ways. $\int_2^5 (x-2)dx =$

   a. Geometrically.

   b. Analytically using a summation.

Finding Total Change

Lesson Goal: Review applications of the definite integral.

6. The traffic flow rate past a certain point on a highway is $q(t) = 30 + 20t - 3t^2$, where q is expressed in hundreds of cars per hour and $t = 0$ at 8:00 AM. How many cars pass be during the time interval from 9:00 to 11:00 AM?

7. The vertical take-off velocity of a helicopter at time t is $v(t) = 0.02t^2 + t$ ft/sec. Find the height of the helicopter 12 seconds after take-off.

Average Value of a Function

8. A frog hops straight up from the ground with an initial velocity of $v_0 = 16$ ft/sec. Taking into account the effect of gravity his height can be expressed as $h(t) = 16t - 16t^2$. Find the frog's average *speed* during one jump.

Antiderivatives

Lesson Goal: Use an antiderivative to evaluate a definite integral.

9. Suppose we want to evaluate: $\int_{-1}^{2} (-3x^2 + 4x)\,dx$.

   a. Let $f'(x) = -3x^2 + 4x$. Can we find $f(x)$?

   b. Use $f(x)$ to evaluate $\int_{-1}^{2} (-3x^2 + 4x)\,dx$.

Thm. **Relationship Between Antiderivatives**: If F and G are both antiderivatives of a function f, then $F(x) = G(x) + C$ for some constant C.

10. *Proof:* Let F and G be antiderivatives of f. Construct the function $H(x) = F(x) - G(x)$

If the theorem is true, then $H(x)$ will be constant.

Assume $H(x)$ **is NOT constant**. Then there exist two points a and b where $H(a) \neq H(b)$.

By the Mean Value Theorem there exists a point $c \in (a,b)$ such that

$H'(c) =$

Since $H(a) \neq H(b)$, then $H(b) - H(a) \neq 0$.

Why can we conclude that $H'(c) \neq 0$?

However $H(x) = F(x) - G(x)$. So, differentiating gives us

$H'(x) =$

for ALL x.

What does this conclusion contradict?

Hence our assumption is false and $H(x)$ must be constant.

11. Find the most general antiderivative.

a. $y = 2x^2 - 6x + 3$

b. $y = 5x^3 - 7x + 8x^{-2}$

c. $y = 4x^{2/3} + 8x^{1/3} - 10$

12. $\int_0^2 3\sqrt{x}\left(x^3 - 5\right)dx =$

13. $\int_0^{\pi/2} \left(x^2 + 2\cos x\right)dx =$

14. $\int_{-1}^{1} (x^2 - 1)(x + 2)dx =$

15. $\int_{2}^{3} \frac{x^2 - 3x + 6}{\sqrt{x}} dx =$

16. $\int_{0}^{\pi/3} (\sec^2 \theta + \sin \theta) d\theta =$

Lesson Goal: Use a definite integral to define an accumulation function.

17. The definite integral can be used to find:

a. ______________________________

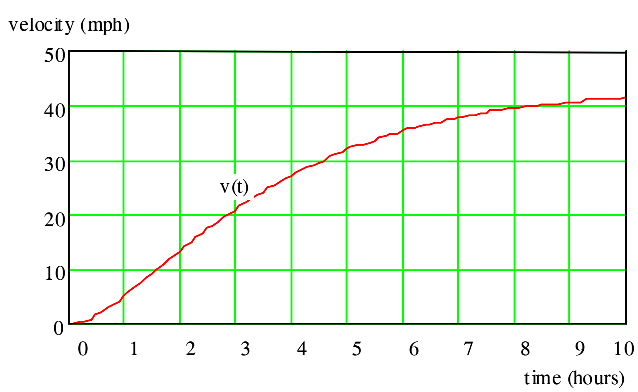
b. ______________________________

c. ______________________________

Constructing the Distance Function

18. Suppose we have a car whose velocity is graphed at right. We want to construct a function to represent the distance traveled at any time x.

a. Let $s(x)$ be the function we are defining. Estimate values for the function at each integer point.



s(0)≈	s(5)≈
s(1)≈	s(6)≈
s(2)≈	s(7)≈
s(3)≈	s(8)≈
s(4)≈	s(9)≈

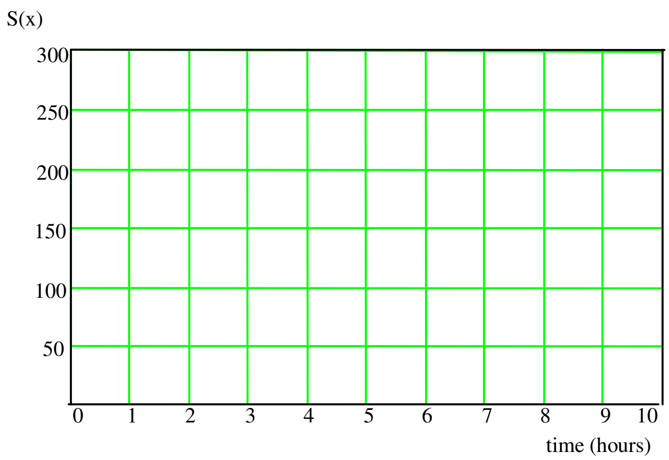
b. Since the definite integral represents the area under the curve, at any point, say $x = 3$ we can say that

$s(3) =$

c. We can consider this as a function then whose values represent the distance traveled, hence

$s(x) =$

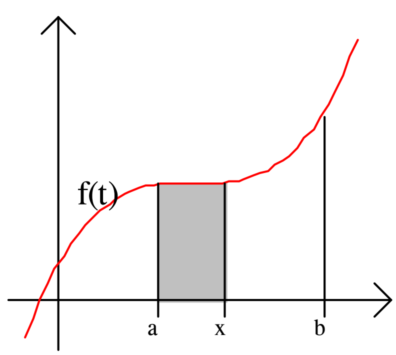
In calculating these values, we are forming a new function that represents the accumulated area under the original function. Plot the values for this new function on a grid.



Def. **Accumulation Function**: Let f be continuous on $[a, b]$ and $x \in [a, b]$. The function

$A(x) = \int_a^x f(t)dt$ is called the accumulation function.

As x moves from a to b, the functional values represent the accumulation of signed area from a to x. In the definition, if $f(t)$ is a velocity function, then we know that $A(x)$ would represent the distance traveled from time a to time x.



19. Suppose we start with the function $f(t) = 2t$. We want to calculate values for the accumulation function $A(x) = \int_0^x 2t\, dt$.

a. Run ACCUMLAT to calculate values for $A(x)$ over the set of points $x \in \{0, 0.2, 0.4, 0.6, \cdots, 4\}$.

ACCUMLAT will ask for the starting point 0, the increment 0.2, and the number of points 20. When the program runs, it will store the x values in List 1 and the $A(x)$ values in List 2. Finally it will graph $f(t)$ and superimpose a scatter plot of $A(x)$.

b. Examine the scatter plot and the values in List 1 and List 2.

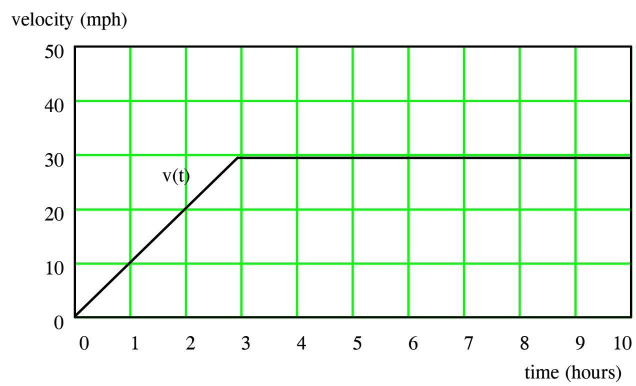
c. Identify the function $A(x)$.

d. What is the relationship between $A(x)=x^2$ and $f(t)=2t$?

e. How would $A(x)$ change if we changed the starting point a?

20. Let $f(t)=\frac{1}{t}$. Run ACCUMLAT and guess the accumulation function from looking at its graph.

21. Suppose we have a car whose velocity is graphed at right.



a. Write an integral expression to define *s(x)*.

b. Use the graph and grid to estimate values for the function.

$s(0)\approx$	$s(5)\approx$
$s(1)\approx$	$s(6)\approx$
$s(2)\approx$	$s(7)\approx$
$s(3)\approx$	$s(8)\approx$
$s(4)\approx$	$s(9)\approx$

c. Use the data values to try to write a formula for $s(x)$

$$s(x)=\Big\{$$

d. Write a formula for $v(t)$:

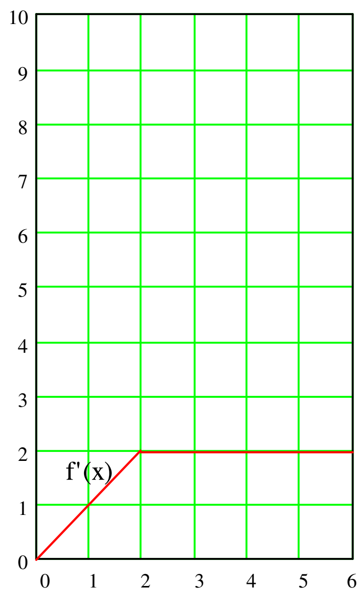
$$v(t)=\Big\{$$

e. What is the relationship between $v(t)$ and $s(x)$?

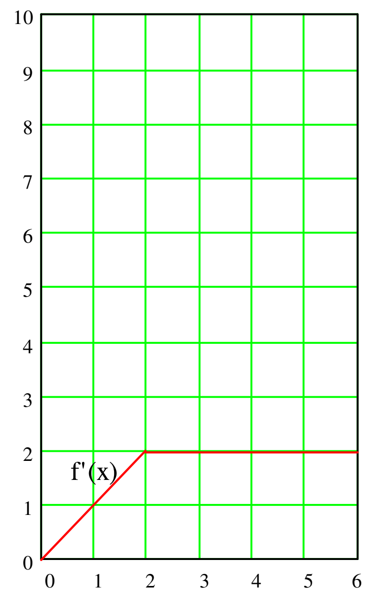
Constructing Antiderivatives Graphically Using Area

Lesson Goal: Use the Second Fundamental Theorem of Calculus to differentiate an accumulation function.

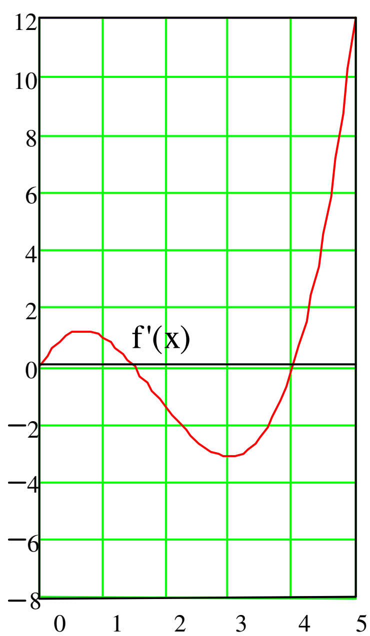
22. Use the graph below of $f'(x)$ and the Second Fundamental Theorem to sketch a graph of $f(x)$ where $f(0)=2$.



23. Do it again for $f(0)=-1$. How do the two antiderivatives differ?



24. Use the graph below $f'(x)$ and the Second Fundamental Theorem to sketch a graph of $f(x)$ where $f(0)=1$.



25. If $F(x)=\int_{-3}^{x}\left(3t^2-8t\right)dt$, find $F'(x)$.

a. First, work with the First Fundamental Theorem of Calculus.

b. Second, work with the Second Fundamental Theorem of Calculus.

26. If $F(x)=\int_5^x\left(t^3-\frac{7}{t}\right)dt$, find $F'(x)$.

27. If $F(x)=\int_0^1\sin(2t)dt$, find $F'(x)$.

28. $\frac{d}{dx}\int_x^7\sqrt{t+4}\,dt$

29. Use the First Fundamental Theorem of Calculus and the Chain Rule to find:

$$\frac{d}{dx}\int_6^{u(x)}f(t)dt$$

30. $\frac{d}{dx}\int_8^{3x^2}\frac{3}{t-7}dt$

Indefinite Integrals

Lesson Goal: Use antiderivatives to solve problems of motion.

31. $\int 5x^8dx=$

32. $\int\left[\frac{4}{x^2}+\frac{3}{x^3}\right]dx=$

33. $\int(x-2)^2(x+8)dx=$

Choose the indefinite integral template and type in the function (*enter*)

Note: User must supply the $+C$ on the end.

Using Indefinite Integrals to Solve Equations of Motion

34. A ball is thrown upward from the ground with a launching velocity of 96 ft/sec. What is the height of the ball after 3 seconds? How high does it go? When does it hit the ground?

35. A man driving an automobile in a straight line at a speed of 80 ft/sec applies the brakes at a certain instant. If the brakes furnish a constant acceleration of -20 ft/sec^2, how far will he go before he stops?

36. A ball is dropped from a height of 8 ft. When it hits the ground, it bounces back up with a speed that is three-fourths of its speed of impact. How high does it go after the first bounce?

Antidifferentiating Using the Chain Rule

Lesson Goal: Use the Chain Rule to find an antiderivative.

37. Differentiate: $\sin(3x)$.

Integrate: $\int 3\cos(3x)dx =$

38. Differentiate: $(x^2 - 1)^3$

Integrate: $\int 6x(x^2 - 1)^2 dx =$

39. Differentiate: $\tan^6(4x)$

Integrate: $\int \sec^2(4x)\tan^5(4x)dx$

40. $\int \sin(5x)dx =$

41. $\int x^2 \sec\left(x^3\right)\tan\left(x^3\right)dx =$

42. $\int 8x(4-x^2)^3 dx =$

43. $\int x^2\left(4-x^3\right)^2 dx =$

44. $\int \frac{\cos x}{\sin^4 x}dx =$

45. $\int \frac{xdx}{\sqrt{x^2-1}} =$

46. $\int \sec^2 x \sqrt[3]{6\tan x}\,dx =$

47. $\int \frac{x^2dx}{\left(x^3+3\right)^2} =$

48. $\int \sqrt{\cot x}\csc^2 x dx =$

49. $\int \cos x(1+\tan x)dx =$

50. $\int \frac{x^2+2x}{\sqrt[3]{x^3+3x^2+1}}dx =$

51. $\int \cot^2\left(\frac{x-5}{2}\right)dx =$

Integrating with Substitutions

Lesson Goal: Use substitution to find an antiderivative.

52. Compare below. Which do you already know how to integrate?

a. $\int \sqrt{x+4}\,dx =$

b. $\int x\sqrt{x^2+4}\,dx =$

c. $\int x\sqrt{x+4}\,dx =$

53. $\int \frac{3x-2}{\sqrt{2x-3}}dx =$

54. $\int_0^1 \left(x^2+2\right)\sqrt{x+1}\,dx =$

55. $\int_{-1}^{1} \frac{x^2}{(x+2)^{\frac{3}{2}}}dx =$

56. Suppose that a company's marginal profit, in millions of dollars, is represented by the function $y=(x-3)\sqrt{x+3}$. Find their total profit over the first two years.

Approximating Area with Trapezoids

Lesson Goal: Use the Trapezoidal Rule to approximate and integral and compare its accuracy with other methods of approximation.

57. Approximate with the Trapezoidal Rule: $\int_1^2 \frac{dx}{x} =$ where $n = 5$.

58. Approximate $\int_1^3 \ln x dx$ with the Trapezoidal Rule using 4 subdivisions.

59. Approximate $\int_1^3 \ln x dx$ with the Midpoint Rule using 4 subdivisions.

60. Use the above two results to get a Simpson's Rule approximation for $\int_1^3 \ln x dx$ using 4 subdivisions.

61. Approximate $\int_1^3 \ln x dx$ with the TI-CAS program INTEGRAL and compare each result with the exact value obtained by using the TI-CAS integral function.