

## CLASSWORK 10 – Parametric, Vector, and Polar Equations

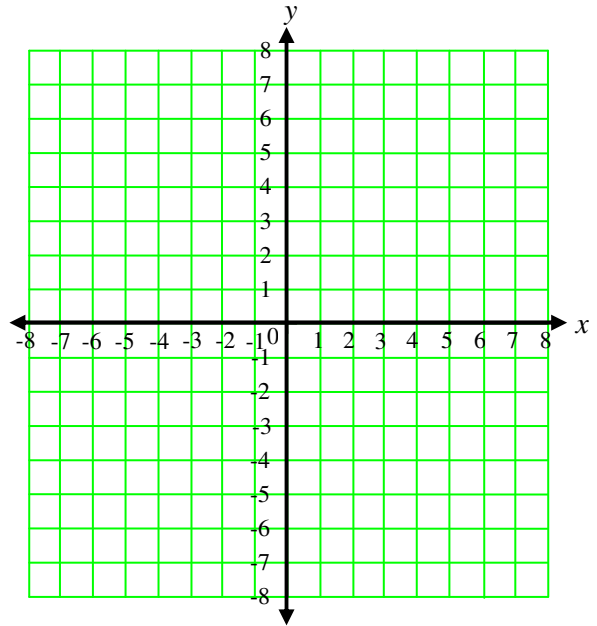
Lesson Goal: Use parametric equations to describe motion.

### Review of Parametric Equations

1.  $x = \sin(2t)$  and  $y = 1 + t$

- a. Make a table and draw graph by hand.  
Indicate direction of motion.

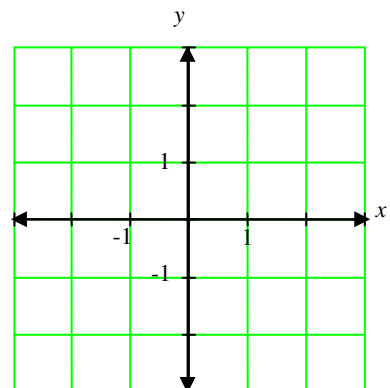
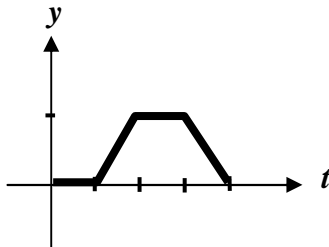
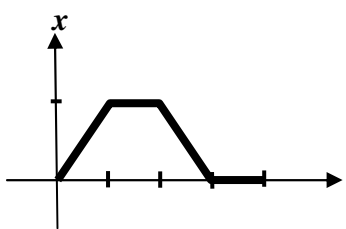

- b. Graph parametrically on TI-CAS.



2. Given the pair of curves  $x_1(t) = t^3 - 2t^2 + t$ ,  $y_1(t) = t$  and  $x_2(t) = 5t$ ;  $y_2(t) = t^3$

- a. Find all collision points. b. Find all points of intersection using TI-CAS.

3. Draw the Cartesian graph indicated by the parametric equations graphed below. Mark the starting point and use arrows to indicate direction of motion.





10. Given a particle that travels with velocity  $\vec{v} = 3\vec{i} + -4\vec{j}$
- Find its displacement over the first 8 seconds.
  - If the particle starts at  $(6, -3)$ , where will it be in 10 seconds.
  - Write parametric equations for its position at any time  $t$ .
11. Find the work done in pushing a car 800 feet across a level parking lot with a force of magnitude 40 pounds directed  $28^\circ$  downward from the horizontal against the back of the car.
12. At a circus performance, a human cannonball is to be fired with an initial speed of  $v_0 = \frac{80\sqrt{10}}{3}$  ft/sec.
- Find the horizontal and vertical components of the initial velocity.
  - If the path that the projectile follows is modeled by the parametric equations  $x(t) = v_0 t \cos \theta$  and  $y(t) = v_0 t \sin \theta - 16t^2$  where  $\theta$  is the launch angle, find the launch angle necessary to land the circus performer on a special cushion located 200 feet downrange.
  - The circus is being held in a large room with a flat ceiling 75 feet high. Can the performer be fired to the cushion without striking the ceiling?

Lesson Goal: Find the speed, velocity, or acceleration of an object whose motion is described parametrically.

**Parametric Motion on a Curve.**

13. A particle moves in the  $xy$ -plane with  $x = t^3 - 9t^2 + 24t$  and  $y = t^2 - 4t + 3$ .
- a. At what time(s) is the particle stopped?
  - b. At what time(s) is the particle moving parallel to the  $x$ -axis or the  $y$ -axis?
  - c. Find the speed of the particle at time  $t = 3$ .
  - d. Find the acceleration vector for the particle when  $t = 2$ .
14. A curve is defined by the parametric equations  $x = t^3 - t$  and  $y = 2t$ .
- a. Write the equation of the tangent line to the curve at the point  $(24, 6)$
  - b. As  $t \rightarrow \infty$ , the curve approaches an asymptote. Find the slope of that asymptote.
  - c. Is the curve a smooth curve?
15. A curve is defined by the parametric equations  $x = 2\sin(\pi t) + 1$  and  $y = t^2 - 4t + 5$  for  $0 \leq t \leq 3.5$
- a. Find the time when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or right at that time?
  - b. There is a point with  $y$ -coordinate 2 through which the particle passes twice. Find the slopes of the lines tangent to the particle's path at that point.
  - c. Find the first time when the line tangent to the path of the particle is vertical. Is the direction of motion of the particle upward or downward at that time?

Lesson Goal: Find the second derivative of a parametric curve and the length of an arc expressed parametrically.

16. Given the curve defined parametrically by  $x = \sqrt{t}$  and  $y = -t^2 + 2t$ .

- a. Express  $\frac{dy}{dx}$  as a function of  $t$ .                      b. Find  $\frac{d^2y}{dx^2}$  in terms of  $t$ .
- c. For what values of  $t$  is the curve concave upward?

17. Consider the Cartesian equation  $y = x^2 + 3x$ .

- a. How would we find the length of the arc from  $x = 0$  to  $x = 2$ ?
- b. Now let's parametrize the arc in the simplest possible way: let  $y = t^2 + 3t$  and  $x = t$ . Use the Parametric Arc Length formula to find the length of the arc.
- c. How are the two formulas equivalent?

18. A particle travels along a path defined by  $x = \sin t$  and  $y = 2 - \cos^2 t$ .

- a. Find the distance the object travels over the time interval  $\left[0, \frac{3\pi}{2}\right]$ .
- b. Convert the parametric equations to Cartesian form and find the two Cartesian points corresponding to  $t = 0$  and  $t = \frac{3\pi}{2}$ . Find the length of the arc between these two Cartesian points.
- c. Why is the value in part (b) not the same as in part (a)?

19. The velocity vector of a particle moving in the  $xy$ -plane has components given by  $\frac{dx}{dt} = \sin(e^t)$  and  $\frac{dy}{dt} = 1 + \cos(2t)$  for  $0 \leq t \leq 2\pi$ .
- a. For  $0 < t < 2\pi$ , find all values of  $t$  at which the line tangent to the path of the particle is horizontal.
- b. Write an equation for the line tangent to the path of the particle at  $t = \frac{\pi}{2}$ .
- c. Find the speed of the particle at  $t = \pi$ .
- d. Find the acceleration vector of the particle at  $t = 1$ .

Lesson Goal: Find the derivative of a vector-valued function and apply it to problems involving velocity and acceleration.

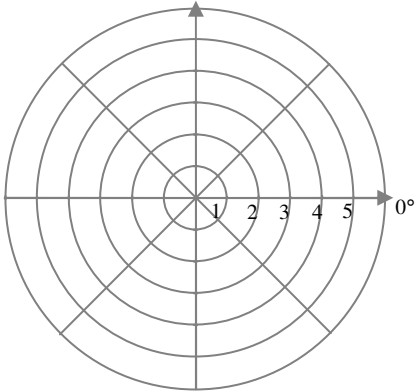
20. Discuss the continuity of the vector-valued function  $\vec{r}(t) = \left\langle \frac{1}{t}, \cos t \right\rangle$ .
21. Given the epicycloid  $\vec{r}(t) = (5\cos t - \cos(5t), 5\sin t - \sin(5t))$ .
- a. Is the curve continuous?
- b. Is it smooth? If not, find all intervals over which it is smooth.
22. Given a particle whose velocity at time  $t$  is described by  $\vec{v}(t) = \langle 6t, \ln(t + 0.5) \rangle$ . At time  $t = 0$  the particle is at the point  $(3, -2)$ .
- a. Find the acceleration vector.
- b. Find an expression for its speed at any time  $t$ .

- c. Find a function  $x(t)$  to describe the  $x$ -coordinate of the position of the particle at any time  $t$ .
- d. Use TI-CAS to find the  $y$ -coordinate of the position of the particle at time  $t = 0.5$ ?
- e. Write the equation of a vector tangent to the path of the particle at time  $t = 0.5$ .
- f. Is the curve increasing or decreasing at  $x = 2$ ?
- g. Is the curve concave upward or concave downward at  $x = 2$ ?
- h. Find any times when the particle is stopped.
- i. Find the distance the particle traveled from  $t = 0$  to  $t = 10$ .
- j. Find any times when the particle is moving straight up or down.
23. A baseball is hit from a position 3 feet above the ground at 100 feet per second at an angle of  $\frac{\pi}{4}$  above the horizontal.
- a. Find the maximum height reached by the baseball.
- b. Will it clear a 10 foot high fence located 300 feet from home plate?

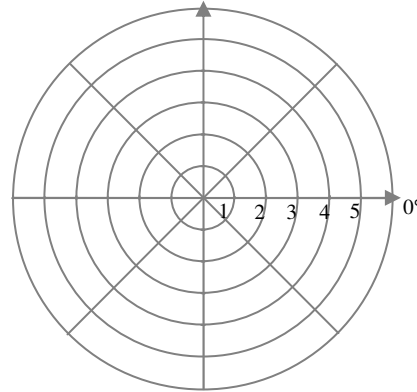
Lesson Goal: Graph a conic in polar form.

**Review of Polar Equations**

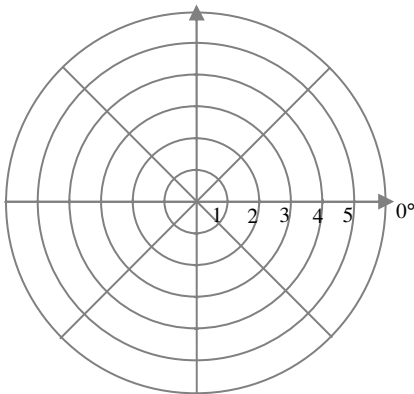
24. Graph  $r = \cos \theta$ .



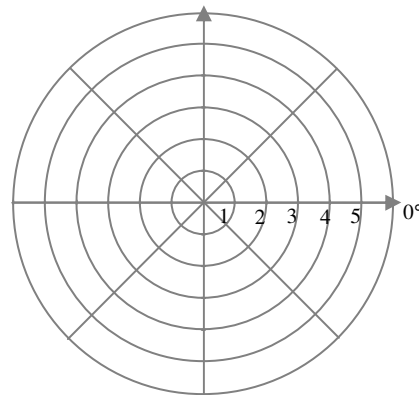
25. Graph  $r = 5 \sin 4\theta$ .



26. Graph  $r = 3 - 4 \sin \theta$ .



27. Graph  $r^2 = 4 \sin 2\theta$  on TI-CAS.



28. Convert  $r = 6 \csc \theta$  to rectangular (or Cartesian) coordinates and identify the curve.

29. Convert  $r = \frac{1}{1 + 2 \cos \theta}$  to rectangular coordinates and identify the curve.

30. Find the set of points in a plane whose distance to a fixed point  $(1, 0)$  is half its distance to a fixed line  $x = 4$ .

31. Identify the conic  $r = \frac{5}{4 - 2 \sin \theta}$ .

32. Sketch a graph of  $r = \frac{5}{4 - 2\sin\theta}$ .

Lesson Goal: Find the polar derivative and the rectangular derivative of a curve in polar form.

**Differentiating Polar Equations**

33. Find the slope of the curve  $r = 1 - \cos\theta$  at  $\theta = \frac{\pi}{2}$ .

34. Write the equation of the tangent line at that point.

35. Find all horizontal and vertical tangent lines to the curve  $r = 1 + \sin\theta$ .

36. Find all maximum and minimum points to the curve  $r = 1 + \sin\theta$ .

37. Find the tangents to the curve  $r^2 = \cos(2\theta)$  at the pole.

38. Find the vertical and horizontal tangent lines to the curve  $r = \sin(2\theta)$ .

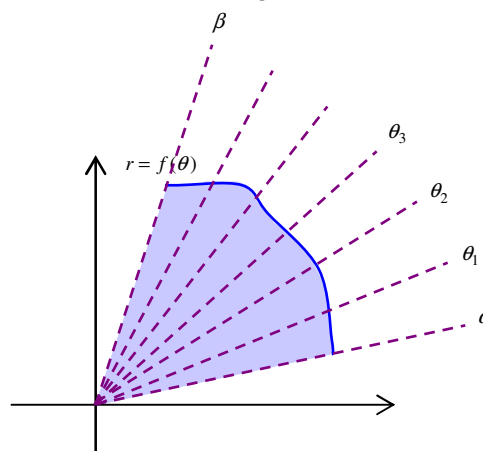
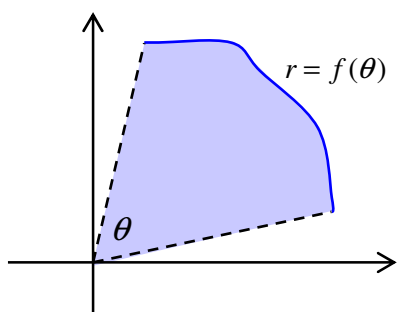
Lesson Goal: Find area on a polar graph.

### Area in Polar Coordinates

**Thm. Sector of a Circle:** The area of a sector of a circle of radius  $r$  with central angle  $\theta$  (in radians) is

$$\text{defined by } A = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} \theta r^2.$$

Suppose a curve is given by  $r = f(\theta)$ , where  $f$  is continuous and non-negative in the interval  $\alpha \leq \theta \leq \beta$ .



First we form a regular partition of the interval  $[\alpha, \beta]$  into  $n$  subintervals:  $\{\alpha = \theta_0, \theta_1, \theta_2, \dots, \theta_n = \beta\}$ .

Note that each subinterval was width  $\Delta\theta = \frac{\beta - \alpha}{n}$  and radius  $f(\theta_i)$ .

So an approximation for the area of one subdivision would be \_\_\_\_\_

Hence an approximation for the total area would be: \_\_\_\_\_.

Finally, the exact area should be the result when we take the limit as  $n \rightarrow \infty$ , or \_\_\_\_\_

**Thm. Area in Polar Coordinates:** If  $r = f(\theta)$  is continuous and non-negative on the interval  $[\alpha, \beta]$ , then the area of the region bounded by the graph of  $r = f(\theta)$  between the radial lines  $\theta = \alpha$  and

$$\theta = \beta \text{ is given by } A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta.$$

Note that the formula is valid as long as the function never changes sign. Because the function is squared in the integral, the result will be positive even when the function is negative.

39. Find the area inside one petal of the polar rose  $r = 5 \cos 2\theta$
- Make a sketch.
  - Determine the limits on the angle that forms one petal.
  - Set up the integral.
  - Evaluate with a calculator.
40. Find the area inside the inner loop of the limaçon  $r = \sqrt{2} - 2 \cos \theta$ .
- Make a sketch.
  - Determine the limits on the angle that forms the inner loop.
  - Set up the integral.
  - Evaluate with a calculator.
41. Find the area inside  $r = \sin \theta$  and  $r = 1 - \sin \theta$ .
- Make a sketch.
  - Find the angle that corresponds to the two points of intersection of the first quadrant region.
  - Set up the integrals. (Remember that the subdivisions are pie shaped regions, not rectangles.)
  - Evaluate with a calculator.