SUPPLEMENTAL PROBLEMS -TRANSCENDENTAL FUNCTIONS

Some problems on this sheet are taken from *Calculus; Single Variable*, second edition by Hughes-Hallett, Gleason, et al. New York, John Wiley & Sons, 1998.

1. For each of the following functions, sketch by hand the direction field on a graph of size $-5 \le x \le 5$ and $-5 \le y \le 5$. Use your TI-89 to check your result. Then try to guess a formula for the general solution. (Remember to set calculator into Mode/Graphing/Diff Eq. Enter an *x* as a *t* and a *y* as *y*1. Use window settings $-5 \le x \le 5$ and $-5 \le y \le 5$.)

   a. $\dfrac{dy}{dx} = x - 1$

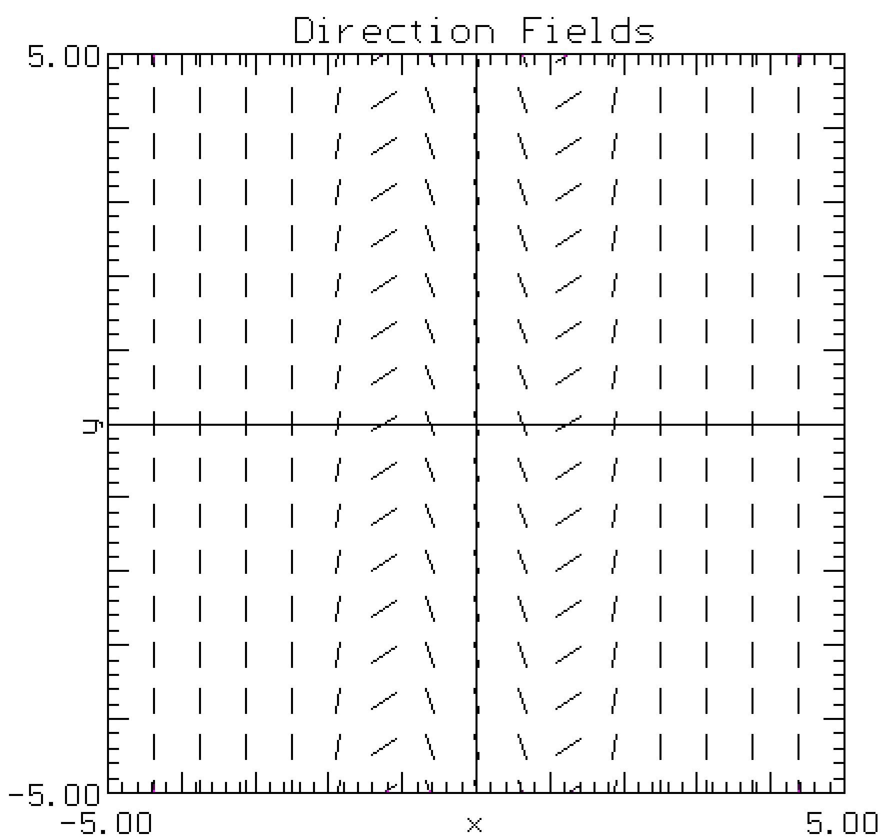
   b. $\dfrac{dy}{dx} = y$

   c. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

2. For the direction field at right:

   a. Draw a specific solution where $f(-2) = 0$.

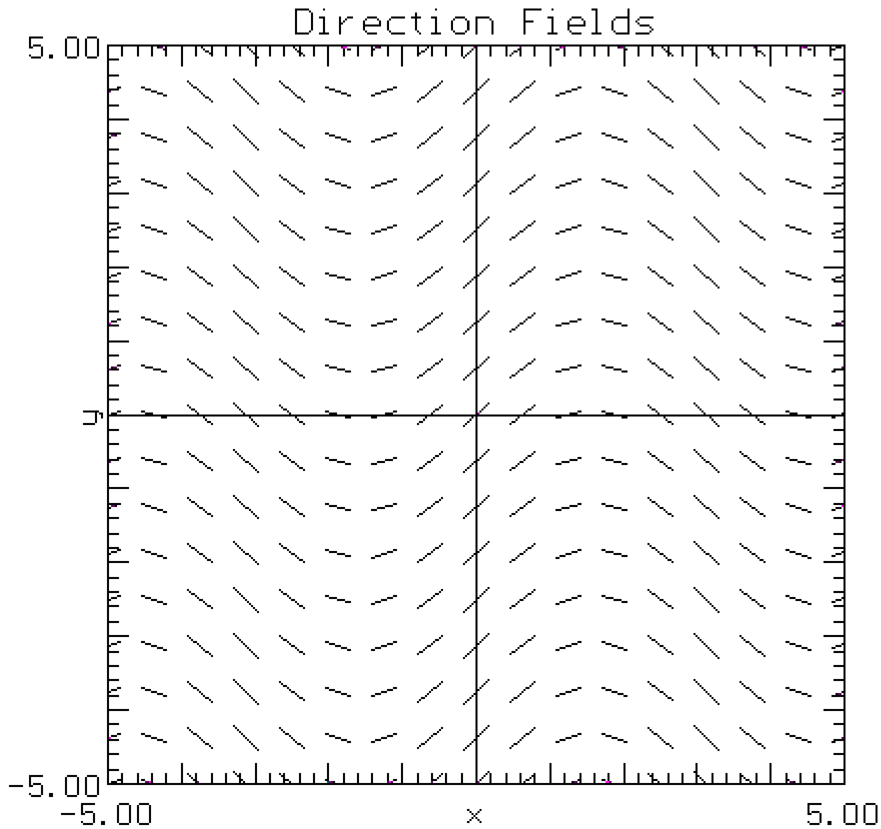
   b. Try to guess a formula for the general solution.



3. For the direction field at right:

   a. Draw a specific solution where $f(-\pi) = 0$.

   b. Try to guess a formula for the general solution.



4. Use Euler's Method to generate points for the function defined by the differential equation $\frac{dy}{dx} = 2x$ with initial condition that $y = 0$ when $x = 0$. Calculate 8 values for y using $\Delta x = 0.5$. Do this problem completely BY HAND - no TI-89 at all.

   Recall that the essence of Euler's Formula is that:
   (new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).
   We know that the starting point is (0, 0). So $x_0 = 0$ and $y_0 = 0$.

   Then $$y_1 = y_0 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0 + (2 * 0) \bullet (0.5) = 0$$
   $$y_2 = y_1 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0 + (2 * 0.5) \bullet (0.5) = 0.5$$
   $$y_3 = y_2 + \left(\frac{dy}{dx}\right) \bullet \Delta x = 0.5 + (2 * 1) \bullet (0.5) = 1.5$$
   Continue from there to generate the remaining points asked for.

5. Given the differential equation $\frac{dP}{dt} = 0.2P$ with initial conditions $P = 1$ when $t = 0$, use Euler's Method to generate values to approximate the function from $t = 0$ to $t = 6$ using $\Delta t = 0.3$.

   Again, Euler's Method says that:
   (new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).

   Expressing this same idea in calculus notation we get:

   $$y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$$

   Substituting in the information from this problem yields:

   $y_{i+1} = y_i + (0.2 \bullet y_i) \bullet 0.3$ with starting point $(0, 1)$ since $P = 1$ when $t = 0$.

   To do this iteratively on the TI-89, type 1 and press ENTER, since $P = 1$ when $t = 0$. That is the starting value. Then type ANS + 0.2•ANS•0.3 and press ENTER. The result will be the value for P when $t = 0.3$. Pressing the ENTER key again will give the value for P when $t = 0.6$. The next press of the ENTER key will give the value for P when $t = 0.9$, etc.

6. The change in the velocity of a body falling at a relatively slow speed over a short distance is given by $\frac{dv}{dt} = g - kv$, where g is the acceleration due to gravity and k is a constant. Let $g = 9.8$ m/sec^2, $k = 0.02$, $\Delta t = 2$, and $v_0 = 0$.

   a. Write a formula for velocity using Euler's Method.

   b. Calculate points for the velocity function over the next 50 seconds. Use the same procedure as in the previous problem. Enter a starting value of 0 in your calculator. Then enter Euler's formula as it relates to this problem, using *ANS* everywhere the v_i variable appears. Now with repeated pressing of the *ENTER* key you should get the appropriate values.

7. Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding air (or water). Let T be the constant temperature of the surroundings, y be the temperature of the object, and t be the time. Then symbolically the above statement says: $\frac{\frac{dy}{dt}}{|y - T|} = k$ or $\frac{dy}{dt} = k|y - T|$.

   a. Use this expression and Euler's Method to write an expression for y_{i+1}.

   b. Suppose a hot object is plunged into a beaker of ice water that is maintained at a constant temperature of 32°F. Let $k = -0.1$, $y_0 = 250°$, and $\Delta t = 0.5$ minutes. Use your TI-89 to produce a dataset of Euler values. Look at the data set to determine the temperature of the object 30 minutes later.

MODE/Graph/Diff Equations *ENTER* *ENTER.*

Set window parameters: $x$max=30 $t$step=0.5

Open the Y= window and enter y_1' as $-0.1 \bullet abs(y1-32)$ Let $t0=0$ and $yi1=250$.

HOME *catalog* *BldData* *ENTER* *APPS* *Data/Matrix Editor/Current* *ENTER.*

For problems # 9 – 20, find the solutions to the differential equation, subject to the given initial conditions.

8. $\frac{df}{dx} = x^2 - 2x + 1$ and $f(0) = 2$.

9. $f'(t) = t^3 - \frac{t^2}{2} - t$ and $f(1) = 1$.

10. $\frac{dP}{dt} = 0.02P$, and $P(0) = 20$.

11. $\frac{dy}{dx} + \frac{y}{3} = 0$, and $y(0) = 10$.

12. $\frac{dP}{dt} = P + 4$, and $P = 100$ when $t = 0$.

13. $\frac{dy}{dt} = 0.5(y - 200)$, and $y = 50$ when $t = 0$.

14. $\frac{dR}{dr} + R = 1$, $R(1) = 0.1$.

15. $\frac{dz}{dt} = te^z$, and the graph passes through the origin.

16. $\frac{dy}{dt} = y^2(1+t)$ and $y = 2$ when $t = 1$.

17. $\dfrac{dz}{dr} = z + zr^2$, and $z(0) = 1$.

For problems # 18 – 19, solve the differential equations. Assume that a and k are constants.

18. $\dfrac{dR}{dt} = kR$.

19. $\dfrac{dP}{dt} = P - a$.

20. When the electromotive force (emf) is removed from a circuit containing inductance and resistance but no capacitors, the rate of decrease of current is proportional to the current. If the initial current is 30 amps but decays to 11 amps after 0.01 seconds, find an expression for the current as a function of time.

21. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For example, this is the case as δ-glucono-lactone changes into gluconic acid.

    a. Write a differential equation satisfied by y, the quantity of δ-glucono-lactone present at time t.

    b. If 100 grams of δ-glucono-lactone is reduced to 54.9 grams in one hour, how many grams will remain after 10 hours?

22. The rate (per foot) at which light is absorbed as it passes through water is proportional to the intensity, or brightness, at that point.

    a. Find the intensity as a function of the distance the light has traveled through the water.

    b. If 50% of the light is absorbed in 10 feet, how much is absorbed in 20 feet? 25 feet?

23. The radioactive isotope carbon-14 is present in small quantities in all life forms. It is constantly replenished until the organism dies, after which it decays to stable carbon-12 at a rate proportional to the amount of carbon-14 present, with a half-life of 5730 years. Suppose $C(t)$ is the amount of carbon-14 present at time t.

    a. Find the value of the constant k in the differential equation $C' = kC$.

    b. In 1988 three teams of scientists found that the Shroud of Turin, which was reputed to be the burial cloth of Jesus, contained 91% of the amount of carbon-14 contained in freshly made cloth of the same material. How old is the Shroud of Turin, according to these data?

24. Warfarin is a drug used as an anticoagulant. After stopping administration of the drug, the quantity remaining in a patient's body decreases at a rate proportional to the quantity remaining. The half-life of Warfarin in the body is 37 hours.

    a. Sketch a rough graph of the quantity, Q, of Warfarin in a patient's body as a function of the time, t, since stopping administration of the drug. Mark the 37 hours on your graph.

    b. Write a differential equation satisfied by Q.

    c. Solve the differential equation.

    d. How many days does it take for the drug level in the body to be reduced to 25% of the original level?

25. The amount of arable land (land that can be used for growing crops) in use increases as the world's population increases. Suppose $A(t)$ represents the total number of hectares of arable land in use in year t. (A hectare is about 2.5 acres.)

    a. Explain why it is plausible that $A(t)$ satisfies the equation $A'(t) = kA(t)$. What assumptions are you making about the world's population and its relation to the amount of arable land used?

    b. In 1950 about 1×10^9 hectares of arable land were in use; in 1980 the figure was 2×10^9. If the total amount of arable land available is thought to be 3.2×10^9 hectares, when does this model predict it is exhausted? (Let $t = 0$ in 1950.)

26. Money in a bank account grows continuously at an annual rate of r (when the interest rate is 5%, $r = 0.05$, and so on). Suppose \$1000 is put into the account in 1970.

    a. Write a differential equation satisfied by M, the amount of money in the account at time t, measured in years since 1970.

    b. Solve the equation.

    c. Sketch a graph of the solution until the year 2000 for interest rates of 5% and 10%.

27. What are the equilibrium solutions for the differential equation $\dfrac{dy}{dt} = 0.2(y-3)(y+2)$?

28. Find the equilibrium solution to the differential equation $\dfrac{dy}{dt} = 0.5y - 250$.

29. Lake Michigan contains 4.9 thousand cubic kilometers of water and has an outflow of 158 cubic kilometers per year. How long would it take for 90% of the pollution to be removed from the lake?

30. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3° F. One hour later, the temperature of the body is 89.0° F. The temperature of the room has been maintained at a constant 68° F.

    a. Assuming the temperature, y, of the body obeys Newton's Law of Cooling, write a differential equation for y.

    b. Solve the differential equation to estimate the time the murder occurred.

31. As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student forgets material is proportional to the difference between the material he or she currently remembers and some positive constant, a.

    a. Let $y = f(t)$ be the fraction of the original material remembered t weeks after the course has ended. Set up a differential equation for y. Your equation will contain two constants. The constant a is less than y for all t.

    b. Solve the differential equation.

    c. Describe the practical meaning (in terms of the amount remembered) of the constants in the solution $y = f(t)$.

32. A small town charity fund drive aims to raise $65,000. Updated current totals are posted in the town square. According to Alfred E. Neuman's law of cooling enthusiasm, the rate at which people contribute to such a drive is proportional to the difference between the current total and the announced target amount. Let $y(t)$ represent the current total, in thousands of dollars, t weeks after the start of the drive. Suppose that after 6 months they have collected $40,000.

    a. Does Neuman's law of cooling enthusiasm sound reasonable? Why or why not?

    b. Express Neuman's law of cooling enthusiasm as a differential equation.

    c. Solve the differential equation.

    d. How long will it take for this town to be within $5,000 of their goal?

33. Suppose that at 1:00 PM one winter afternoon, there is a power failure at your house in Wisconsin, and that your heat does not work without electricity. When the power goes out, it is 68° F in your house. At 10:00 PM, it is 57° F in the house, and you notice that it is 10° F outside.

    a. Assuming that the temperature y in your home obeys Newton's Law of Cooling, write the differential equation satisfied by y.

    b. Use Euler's Method to write a recursive formula for y. Let the change in t be 1 hour.

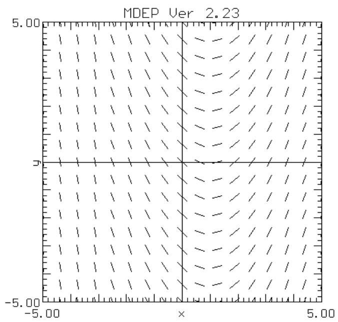
c. Next you need to find the constant of proportionality. Use a trial and error process with your TI-89. Start with a guess of $k = -0.02$. See what temperature that gives you at 10:00 PM. Is the value you got more or less than the $57°$ it is supposed to be? Revise your guess for k until you come reasonably close to $57°$.

d. What will be the temperature in the house when you get up at 7:00 AM the next morning? Should you worry about your water pipes freezing?

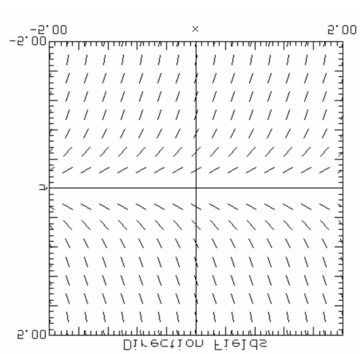
e. What assumption did you have to make in part a. about the temperature outside? Knowing that this assumption is probably incorrect, would you revise your estimate up or down? Why?

Answers

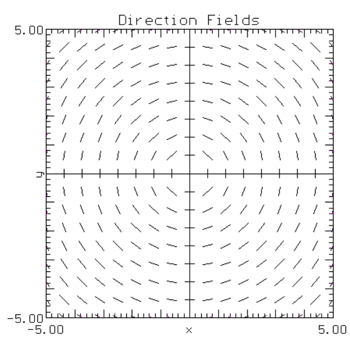
1. a. $y = \frac{1}{2}x^2 - x + c$



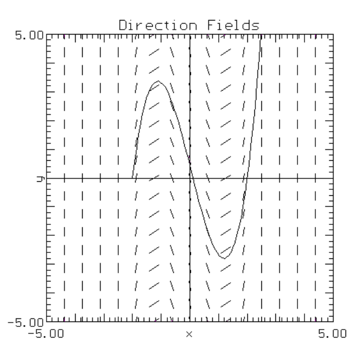
b. $y = e^x + c$



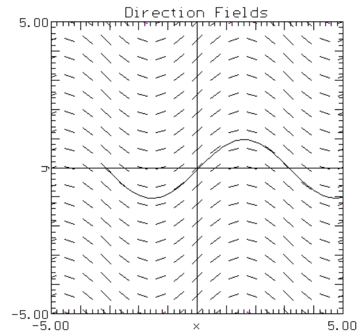
c. $x^2 + y^2 = r^2$



2. $y = x^3 - 4x$



3. $y = \sin(x)$



4.

x	y
0	0
0.5	0
1	0.5
1.5	1.5
2	3
2.5	5
3	7.5
3.5	10.5

5. $P_{i+1} = P_i + \frac{dP}{dt}(\Delta t)$

t	P
0	1
0.3	1.06
0.6	1.1236
0.9	1.1910
1.2	1.2625
. . .	

6. a. $v_{i+1} = v_i + (g - kv_i)(\Delta t)$
$= v_i + (9.8 - 0.02v_i)(2)$

t	v
0	0
2	19.6
4	38.416
6	56.479
8	73.820
. . .	

7. a. $y_{i+1} = y_i + (k|y_i - T|)(\Delta t)$

b. 42.043°F

t	y
0	250
0.5	239.1
1	228.75
1.5	218.91
2	209.56
.	

8. $f(x) = \frac{1}{3}x^3 - x^2 + x + 2$

9. $f(t) = \frac{1}{4}t^4 - \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{17}{12}$

10. $P = 20e^{0.02t}$

11. $y = 10e^{-x/3}$

12. $P = 104e^t - 4$

13. $y = 200 - 150e^{t/2}$

14. $R(r) = 1 - 0.9e^{1-r}$

15. $z = -\ln\left(1 - \frac{t^2}{2}\right)$

16. $y = \frac{-2}{t^2 + 2t - 4}$

17. $z(r) = e^{r + r^3/3}$

18. $R = Ae^{kt}$

19. $P = a + Ae^t$

20. $C(t) = 30e^{-100.33t}$

21. a. $\frac{\frac{dy}{dt}}{y} = k$

b. $y = 100e^{t\ln 0.549}$; $y = 0.249$

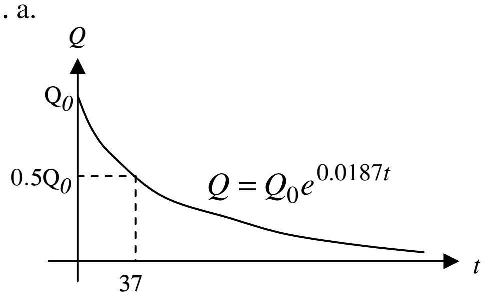
22. a. $\frac{\frac{dI}{ds}}{I} = k$; $I = I_0 e^{ks}$ Where I is the intensity of light *remaining*.

b. $I = I_0 e^{((1/10)\ln 0.5)s}$; At $s = 20$, $\frac{I}{I_0} = 0.25$. Hence 75% has been absorbed.
At $s = 25$, $\frac{I}{I_0} = 0.177$. Hence 82.3% of the light has been absorbed.

23. a. $k = (1/5730)\ln 0.5$

b. $0.91 = e^{(t/5730)\ln 0.5}$; 779.6 years

24. a.



b. $\dfrac{\frac{dQ}{dt}}{Q} = k$

c. $Q = Ce^{kt}$; $Q = Ce^{((1/37)\ln 0.5)t}$

d. 3 days

25. a. Since population satisfies a differential equation of the form $P'(t) = kP(t)$, so should the amount of arable land. You are assuming that the amount of arable land used is proportional to the population.

b. 2001

26. a. $\dfrac{\frac{dM}{dt}}{M} = r$

b. $M = 1000e^{rt}$

27. $\{-2, 3\}$

28. $\{500\}$

29. 71 years

30. a. $\dfrac{\frac{dy}{dt}}{y - T} = k$

b. $y = 68 + 22.3e^{t \bullet \ln(21/22.3)}$ where $t = 0$ at the time the body is *found*. The murder occurred at 3:44 AM.

31. a. $\dfrac{\frac{dy}{dt}}{y - a} = k$

b. $y = (1 - a)e^{kt} + a$

c. a is the amount remembered over a long period of time.
k is the rate at which the material is forgotten.

32. b. $\dfrac{\frac{dy}{dt}}{y - 65{,}000} = k$

c. $y = 65{,}000 - 65{,}000e^{-0.15925t}$

d. about 16 months

33. b. $y_{i+1} = y_i + k(y - 10) \bullet 1$

c. $k = -0.023095$

d. about 48°