

## 99 DEFINITIONS AND THEOREMS YOU'LL WISH YOU KNEW FOR THE AP EXAM

**Remember to review your sheet of trigonometry formulas also.**

All of the definitions and theorems below are required for the AP Calculus Examination. Particular attention should be given those marked with the ❖ symbol.

1. Def. **Limit:**  $\lim_{x \rightarrow c} f(x) = L$  means that  $f(x)$  becomes increasingly close to  $L$  as  $x$  gets closer to  $c$ .

❖ 2. Thm. **Limit Existence:** If  $f(x)$  is a function and  $c$  and  $L$  are real numbers, then  $\lim_{x \rightarrow c} f(x) = L$  if and only if  $\lim_{x \rightarrow c^-} f(x) = L$  and  $\lim_{x \rightarrow c^+} f(x) = L$ .

3. Thm. **Properties of Limits:** Let  $b$  and  $c$  be real numbers, let  $n$  be a positive integer, and let  $f$  and  $g$  be functions whose limit at  $c$  exists. Then

a. Constant Function:  $\lim_{x \rightarrow c} b = b$ .

b. Scalar Multiple:  $\lim_{x \rightarrow c} [b \cdot f(x)] = b \cdot \lim_{x \rightarrow c} f(x)$ .

c. Sum or Difference:  $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x)$ .

d. Product:  $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$ .

e. Quotient:  $\lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$  provided  $\lim_{x \rightarrow c} g(x) \neq 0$ .

f. Power:  $\lim_{x \rightarrow c} [f(x)]^n = \left[ \lim_{x \rightarrow c} f(x) \right]^n$ .

4. Thm. **Squeeze Law:** If  $h(x) \leq f(x) \leq g(x)$  for all  $x$  in an open interval containing  $c$ , except possibly at  $c$  itself, and if  $\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$ , then  $\lim_{x \rightarrow c} f(x) = L$ .

5. Thm. **L'Hôpital's Rule:** If  $\lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right]$  results in the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , then

$\lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right] = \lim_{x \rightarrow c} \left[ \frac{f'(x)}{g'(x)} \right]$  provided the latter limit is of a determinate form.

6. Thm. **Trig Limit for the Sine:**  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  for  $x$  measured in *radians*.

7. Thm. **Trig Limit for the Cosine:**  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$ .
8. Def. **e:**  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$  or  $e = \lim_{n \rightarrow 0} (1 + n)^{\frac{1}{n}}$ .
- ❖ 9. Def. **Continuity at a point:** A function  $f(x)$  is continuous at  $c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ .
10. Def. **Continuity on an open interval:** A function is continuous on an open interval  $(a, b)$  if it is continuous at each point in the interval.
11. Def. **Continuity on a closed interval:** A function is continuous on a closed interval  $[a, b]$  if it is continuous on  $(a, b)$ ,  $\lim_{x \rightarrow a^+} f(x) = f(a)$ , and  $\lim_{x \rightarrow b^-} f(x) = f(b)$ .
12. Def. **Discontinuity:** A function  $f$  is discontinuous at  $c$  if  $f$  is defined on an open interval containing  $c$  (except possibly at  $c$ ) and  $f$  is not continuous at  $c$ .
13. Def. **Removable discontinuity:** A discontinuity at  $x = c$  is called removable if  $f$  can be made continuous by appropriately defining (or redefining) the single point  $x = c$ .
14. Thm. **Properties of Continuity:** If  $b$  is a real number and  $f(x)$  and  $g(x)$  are continuous at  $x = c$ , then each of the following functions are also continuous at  $c$ :
- a. Scalar Multiple:  $b \cdot f(x)$
  - b. Sum or Difference:  $f(x) \pm g(x)$
  - c. Product:  $f(x) \cdot g(x)$
  - d. Quotient:  $\frac{f(x)}{g(x)}$  provided  $g(c) \neq 0$
15. Thm. **Continuity for Polynomial and Rational Functions:** Polynomial functions are everywhere continuous. Rational functions are continuous on their domain.
16. Thm. **Continuity for Composite Functions:** If  $g(x)$  is continuous at  $c$  and  $f(x)$  is continuous at  $g(c)$ , then the composite function given by  $f(g(x))$  is continuous at  $x = c$ .
- ❖ 17. Thm. **Intermediate Value Theorem:** If  $f(x)$  is continuous on  $[a, b]$  and  $k$  is any number between  $f(a)$  and  $f(b)$ , then there is at least one number  $c$  between  $a$  and  $b$  such that  $f(c) = k$ .
- Geometric Application:* Under the given conditions, if  $f(a)$  and  $f(b)$  have opposite signs, then there is a point in the open interval where the graph crosses the  $x$ -axis.

- ❖ 18. Def. **Difference quotient:** The expression  $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$  is called a difference quotient and represents the average rate of change of  $f(x)$  over the interval  $[x_1, x_2]$ .

Alternate notation for a difference quotient is  $\frac{f(x+h) - f(x)}{h}$  or  $\frac{f(x+\Delta x) - f(x)}{\Delta x}$ .

19. Def. **Average velocity:** The average velocity of an object over an interval of time is the net change in position during the interval divided by the change in time. For a function  $s(t)$ , that is  $v = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ .

- ❖ 20. Def. **Derivative:**  $f'(x_1) = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$  (if it exists) is called the derivative of  $f(x)$  at  $x_1$  and represents the instantaneous rate of change of  $f(x)$  at the point  $x_1$ .

21. Def. **Instantaneous velocity:** The instantaneous velocity of an object at time  $t_1$  is given by the limit of the average velocity as  $t_2$  approaches  $t_1$ . For the function  $s(t)$ , that is

$$v(t_1) = \lim_{t_2 \rightarrow t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

- ❖ 22. Def. **Derivative, alternate forms:**  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  (if it exists) or  $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$  (if it exists).

23. Def. **Right-hand derivative:** The right-hand derivative of  $f(x)$  is  $\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$ , provided the limit exists.

24. Def. **Left-hand derivative:** The left-hand derivative of  $f(x)$  is  $\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}$ , provided the limit exists.

- ❖ 25. Thm. **Differentiability and Continuity:** If a curve is differentiable at a point  $x = c$ , then it is continuous at  $x = c$ .

26. Def. **Local linearity:** A curve is called locally linear over an interval when zooming in on the curve causes it to look like a straight line.

27. Thm. **Local Linearity and Differentiability:** If a curve is locally linear at a point  $x = c$  and the tangent line is not vertical there, then the function is differentiable at  $x = c$ .

28. Def. **Slope of a curve:** The slope of a curve at a point is the slope of the tangent line at the point.

29. Def. **Normal line:** A normal line to a curve at a point is a line perpendicular to the tangent line at the point.
- ❖ 30. Def. **Differential of  $y$ :** Let  $y = f(x)$  represent a differentiable function and let  $dx$  be any nonzero change in  $x$ . Then the differential of  $y$ , written  $dy$  is given by  $dy = f'(x)dx$ .
31. Def. **Linear approximation of  $\Delta y$ :**  $dy$  is called the linear approximation of the actual increment,  $\Delta y$ .
- ❖ 32. Def. **Linear approximation of  $f(x)$ :** The expression  $f(x) \approx f'(a)(x-a) + f(a)$  is called the linear approximation to  $f(x)$  near  $x = a$ . It is just another name for the tangent line.
- ❖ 33. Thm. **Rolle's Theorem:** If  $f$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $f(a) = f(b)$ , then there is at least one number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .
- ❖ 34. Thm. **Mean Value Theorem:** If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists a number  $c$  in  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the tangent to the curve is the same as the slope of the line joining the endpoints.

**Application:** Under the given conditions, there is a point in the open interval where the instantaneous rate of change is the same as the average rate of change on the interval. If the function is a position function, then there is a point in the open interval where the instantaneous velocity is the same as the average velocity on the interval.

- ❖ 35. Thm. **Properties of Derivatives:**
- g. If  $y = c$ , then  $y' = 0$ .
  - h. If  $y = c \cdot f(x)$ , then  $y' = c \cdot f'(x)$ .
  - i. If  $y = f(x) \pm g(x)$ , then  $y' = f'(x) \pm g'(x)$ .
  - j. If  $y = x^n$ , then  $y' = nx^{n-1}$ .
36. Thm. **Product Rule:** If  $f(x)$  and  $g(x)$  are differentiable functions at  $x$ , then
- $$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$
37. Thm. **Quotient Rule:** If  $f(x)$  and  $g(x)$  are differentiable functions at  $x$ , and  $g(x) \neq 0$ , then
- $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$
- ❖ 38. Thm. **Chain Rule:** If  $y = f(g(x))$  is a differentiable function of  $g(x)$ , and  $g(x)$  is a differentiable function of  $x$ , then  $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ .

- ❖ 39. Thm. **Alternate Form of Chain Rule:** If  $y = f(u)$  is a differentiable function of  $u$ , and

$$u = g(x) \text{ is a differentiable function of } x, \text{ then } \frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}.$$

- ❖ 40. Thm. **Derivatives of the Trigonometric Functions:**

a. **Sine:** If  $y = \sin x$ , then  $y' = \cos x$ .

b. **Cosine:** If  $y = \cos x$ , then  $y' = -\sin x$ .

c. **Tangent:** If  $y = \tan x$ , then  $y' = \sec^2 x$ .

d. **Cotangent:** If  $y = \cot x$ , then  $y' = -\csc^2 x$ .

e. **Secant:** If  $y = \sec x$ , then  $y' = \sec x \tan x$ .

f. **Cosecant:** If  $y = \csc x$ , then  $y' = -\csc x \cot x$ .

41. Thm. **Absolute Value Rule:** If  $y = |x|$ , then  $y' = \frac{|x|}{x}$ .

42. Def. **Exponential function:**  $y = a^x$  where  $a > 0$  and  $a \neq 1$  is called the exponential function.

43. Def. **Inverse trigonometric functions:**

a. **Inverse Sine:**  $y = \arcsin x$  means  $x = \sin y$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .

b. **Inverse Cosine:**  $y = \arccos x$  means  $x = \cos y$  where  $0 \leq y \leq \pi$ .

c. **Inverse Tangent:**  $y = \arctan x$  means  $x = \tan y$  where  $-\frac{\pi}{2} < y < \frac{\pi}{2}$ .

d. **Inverse Cotangent:**  $y = \text{arccot } x$  means  $x = \cot y$  where  $0 < y < \pi$ .

e. **Inverse Secant:**  $y = \text{arcsec } x$  means  $x = \sec y$  where  $0 \leq y \leq \pi$  and  $y \neq \frac{\pi}{2}$ .

f. **Inverse Cosecant:**  $y = \text{arccsc } x$  means  $x = \csc y$  where  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$  and  $y \neq 0$ .

- ❖ 44. Thm. **Derivatives of the Inverse Trigonometric Functions:**

a. **Arcsine:** If  $y = \arcsin x$ , then  $y' = \frac{1}{\sqrt{1-x^2}}$ .

b. **Arccosine:** If  $y = \arccos x$ , then  $y' = \frac{-1}{\sqrt{1-x^2}}$ .

c. **Arctangent:** If  $y = \arctan x$ , then  $y' = \frac{1}{1+x^2}$ .

d. **Arccotangent:** If  $y = \text{arccot } x$ , then  $y' = \frac{-1}{1+x^2}$ .

e. **Arcsecant:** If  $y = \operatorname{arcsec} x$ , then  $y' = \frac{1}{|x|\sqrt{x^2 - 1}}$ .

f. **Arccosecant:** If  $y = \operatorname{arccsc} x$ , then  $y' = \frac{-1}{|x|\sqrt{x^2 - 1}}$ .

❖ 45. Thm. **Derivative of an Exponential Function:** If  $y = a^x$ , then  $y' = a^x \ln a$ .

❖ 46. Thm. **Derivative of the Natural Exponential Function:** If  $y = e^x$ , then  $y' = e^x$ .

47. Def. **Logarithm function:** If  $y = \log_a x$ , then  $a^y = x$ , where  $a > 0$ ,  $a \neq 1$ , and  $x > 0$ .  
Notation:  $\ln x = \log_e x$  and  $\log x = \log_{10} x$ .

48. Thm. **Properties of the Natural Logarithm.**

a. The domain of  $y = \ln x$  is  $(0, \infty)$  and the range is  $(-\infty, \infty)$ .

b. The function is continuous, increasing, one-to-one.

c. The graph is concave downward on its entire domain.

d.  $\lim_{x \rightarrow \infty} \ln x = \infty$  and  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .

e.  $\log_b 1 = 0$ .

f.  $\ln pq = \ln p + \ln q$ .

g.  $\ln \frac{p}{q} = \ln p - \ln q$ .

h.  $\ln p^n = n \ln p$ .

i.  $\log_a x = \frac{\log_b x}{\log_b a}$ .

j.  $e^{\ln x} = x$ .

k.  $\ln e^x = x$ .

❖ 49. Thm. **Derivative of a Logarithm.** If  $y = \log_a x$ , then  $y' = \frac{1}{x \ln a}$ .

❖ 50. Thm. **Derivative of the Natural Logarithm.** If  $y = \ln x$ , then  $y' = \frac{1}{x}$ .

❖ 51. Thm. **Extreme Value Theorem:** If  $f(x)$  is continuous on  $[a, b]$ , then  $f$  has both a maximum and minimum value on the interval.

- ❖ 52. Def. **Critical point:** If  $f$  is defined at  $c$ , and  $f'(c)=0$  or  $f'(c)$  is undefined, then  $c$  is called a critical point of  $f$ .
- 53. Thm. **Finding Relative Extrema:** If  $f$  has a relative maximum or minimum at  $x=c$ , then  $c$  is a critical number of  $f$ .
- 54. Def. **Horizontal asymptote:** A horizontal asymptote is a line  $y=L$  such that  $\lim_{x \rightarrow \infty} f(x) = L$  or  $\lim_{x \rightarrow -\infty} f(x) = L$ .
- 55. Def. **Vertical asymptote:** A vertical asymptote is a line  $x=a$  such that  $\lim_{x \rightarrow a} f(x) = \infty$ .
- 56. Def. **Increasing function:** A function  $f(x)$  is increasing on an interval if for any two numbers  $x_1$  and  $x_2$  in the interval,  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$ .
- 57. Def. **Decreasing function:** A function  $f(x)$  is decreasing on an interval if for any two numbers  $x_1$  and  $x_2$  in the interval,  $x_1 < x_2$  implies  $f(x_1) > f(x_2)$ .
- ❖ 58. Thm. **Interpreting the Derivative:** Let  $f(x)$  be a function that is differentiable on the open interval  $(a, b)$ . Then:
  - a. If  $f'(x) > 0$  for all  $x$  in  $(a, b)$ , then  $f(x)$  is increasing on  $(a, b)$ .
  - b. If  $f'(x) < 0$  for all  $x$  in  $(a, b)$ , then  $f(x)$  is decreasing on  $(a, b)$ .
  - c. If  $f'(x) = 0$  for all  $x$  in  $(a, b)$ , then  $f(x)$  is constant on  $(a, b)$ .
- ❖ 59. Thm. **First Derivative Test for Local Extrema:** Let  $c$  be a critical point of the function  $f$  that is continuous on an open interval. If  $f$  is differentiable on the interval, except possibly at  $c$ , then  $f(c)$  can be classified as follows.
  - a. If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a relative minimum at  $c$ .
  - b. If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a relative maximum at  $c$ .
  - c. If  $f'$  does not change signs at  $c$ , then  $f$  has neither a relative maximum nor a relative minimum at  $c$ .
- ❖ 60. Def. **Concave upward:** The graph of a differentiable function  $f$  is concave upward on an interval if  $f'$  is increasing on the interval.
- ❖ 61. Def. **Concave downward:** The graph of a differentiable function  $f$  is concave downward on an interval if  $f'$  is decreasing on the interval.
- ❖ 62. Thm. **Test for Concavity:** Let  $f(x)$  be a function whose second derivative exists on an open interval  $(a, b)$ . Then:
  - a. If  $f''(x) > 0$  for all  $x$  in  $(a, b)$ , then the graph of  $f(x)$  is concave upward.
  - b. If  $f''(x) < 0$  for all  $x$  in  $(a, b)$ , then the graph of  $f(x)$  is concave downward.

- ❖ 63. Def. **Inflection point:** If the graph of  $f$  changes concavity at  $c$  and there exists a tangent line to the curve at  $c$ , then  $c$  is called an inflection point of  $f$ .
- ❖ 64. Thm. **Finding Inflection Points with the Second Derivative:** If  $c$  is a point of inflection of the graph of  $f$ , then either  $f''(c) = 0$  for  $f''$  is undefined at  $x = c$ .
- ❖ 65. Thm. **Second Derivative Test for Relative Extrema:** Let  $f$  be a function such that  $f'(c) = 0$  and the second derivative of  $f$  exists on some open interval containing  $c$ .
  - a. If  $f''(c) > 0$ , then  $f(c)$  is a relative minimum.
  - b. If  $f''(c) < 0$ , then  $f(c)$  is relative maximum.
  - c. If  $f''(c) = 0$  or does not exist, the test fails.
- ❖ 66. Def. **Riemann sum:** Let  $f$  be defined on the closed interval  $[a, b]$  which is partitioned by the set  $\{a = x_0, x_1, x_2, \dots, x_n = b\}$ . If  $c_i \in [x_{i-1}, x_i]$  and  $\Delta x_i = x_i - x_{i-1}$ , then the sum
 
$$\sum_{i=1}^n f(c_i) \Delta x_i$$
 is called a Riemann sum of  $f$  for the given partition.
- ❖ 67. Def. **Definite integral:** If  $f$  is defined on the interval  $[a, b]$  and the limit of the Riemann sum
 
$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$
 exists, then this limit is called the definite integral of  $f$  on  $[a, b]$  and is denoted by  $\int_a^b f(x) dx = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$ . (The values  $a$  and  $b$  are called the lower and upper limits of the integral, respectively.)
- 68. Def. **Existence of the definite integral:** If  $f$  is a continuous function on  $[a, b]$ , then the limits as  $n \rightarrow \infty$  of both the lower and upper sums exist and are equal to each other.
- ❖ 69. Thm. **Continuity and Integrability:** If a function  $f$  is continuous on the closed interval  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .
- 70. Thm. **Integration of an Odd Function:** If  $f$  is an odd function which is integrable on  $[-a, a]$ , then  $\int_{-a}^a f(x) dx = 0$ .
- 71. Thm. **Integration of an Even Function:** If  $f$  is an even function which is integrable on  $[-a, a]$ , then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ .

72. Thm. **Properties of the Definite Integral:** Given that  $f$  and  $g$  are two integrable functions on  $[a, b]$ .

a. **At a point:**  $\int_a^a f(x)dx = 0$ .

b. **Over a reverse interval:**  $\int_b^a f(x)dx = -\int_a^b f(x)dx$ .

c. **Two Adjacent Intervals:**  $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

d. **Constant Times a Function:**  $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$ .

e. **Sum or Difference of Functions:**  $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ .

73. Thm. **Comparing Definite Integrals:** If  $f$  and  $g$  are both integrable functions on  $[a, b]$  and  $f(x) \leq g(x)$  for  $a \leq x \leq b$ , then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ .

❖ 74. Thm. **Trapezoidal Rule:** If  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(x)dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

75. Def. **Antiderivative:** If  $f'(x)$  is the derivative of  $f(x)$ , then  $f(x)$  is called an antiderivative of  $f'(x)$ .

76. Thm. **Relationship Between Antiderivatives:** If  $F$  and  $G$  are both antiderivatives of a function  $f$ , then  $F(x) = G(x) + C$  for some constant  $C$ .

77. Thm. **Distance versus Displacement:** If the continuous function  $v(t)$  represents the velocity of a function over an interval of time  $[a, b]$ , then:

a. Displacement (or change in position) =  $\int_a^b v(t)dt$ .

b. Total Distance Traveled =  $\int_a^b |v(t)|dt$

❖ 78. Thm. **Fundamental Theorem of Calculus:** If a function  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is an antiderivative of  $f$  on the interval  $[a, b]$ , then  $\int_a^b f(x)dx = F(b) - F(a)$ .

**Restatement:** The total change in a function is the definite integral of its rate of change.

79. Def. **Accumulation function:** Let  $f$  be continuous on  $[a, b]$  and  $x \in [a, b]$ . The function

$$A(x) = \int_a^x f(t)dt \text{ is called the accumulation function.}$$

❖ 80. Thm. **Second Fundamental Theorem of Calculus:** If  $f$  is continuous on an open interval containing  $a$ , then for every  $x$  in the interval where  $F(x) = \int_a^x f(t)dt$ ,  $F(x)$  is an antiderivative of  $f(x)$ .

81. Def. **Indefinite integral:** The expression  $\int f(x)dx$  is called the indefinite integral and is used to represent the family of all antiderivatives of  $f(x)$ .

❖ 82. Thm. **Properties of the Indefinite Integral:**

a.  $\int dx = x + C$ .

b. **Constant Times a Function:**  $\int k \cdot f(x)dx = k \cdot \int f(x)dx$ .

c. **Sum or Difference of Functions:**  $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$ .

❖ 83. Thm. **Integrals of the Trigonometric Functions:**

a. **Sine:**  $\int \sin x dx = -\cos x + C$ .

b. **Cosine:**  $\int \cos x dx = \sin x + C$ .

c. **Tangent:**  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$ .

d. **Cotangent:**  $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$ .

e. **Secant:**  $\int \sec x dx = \ln|\sec x + \tan x| + C$ .

f. **Cosecant:**  $\int \csc x dx = -\ln|\csc x + \cot x| + C$ .

❖ 84. Thm. **Integration with the Inverse Trigonometric Functions:**

a. **Arcsine:**  $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ .

b. **Arctangent:**  $\int \frac{1}{1+x^2} dx = \arctan x + C$ .

c. **Arcsecant:**  $\int \frac{1}{x\sqrt{x^2-1}} = \text{arc sec}|x| + C.$

85. Thm. **Integrals of Exponential Functions:**

a.  $\int e^x dx = e^x + C.$

b.  $\int a^x dx = \frac{1}{\ln a} a^x + C.$

86. Thm. **Integration of  $\frac{1}{x}$ :**  $\int \frac{1}{x} dx = \ln|x| + C$  for  $x \neq 0.$

❖ 87. Thm. **Area Between Two Curves:** If  $f$  and  $g$  are continuous on  $[a, b]$  and  $g(x) \leq f(x)$  for all  $x$  in  $[a, b]$ , then the area of the region bounded by the graphs of  $f$  and  $g$  and the vertical lines  $x = a$  and  $x = b$  is  $\int_a^b [f(x) - g(x)] dx.$

88. Thm. **Area Under a Curve:** Let  $f$  be a continuous, function on the interval  $[a, b]$ . The area of the region bounded by the graph of  $f$ , the  $x$ -axis, and the vertical lines  $x = a$  and  $x = b$  is found by:

$$\text{area} = \int_a^b |f(x)| dx.$$

❖ 89. Def. **Average value of a function:** If  $f$  is a continuous function on  $[a, b]$ , then the average value of  $f$  on  $[a, b]$  is found by  $\bar{y} = \frac{1}{b-a} \int_a^b f(x) dx.$

❖ 90. Thm. **Average Value Theorem for Integrals:** If  $f(x)$  is continuous on  $[a, b]$ , then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$  (Note that in the Larson text this theorem is referred to as the Mean Value Theorem for Integrals.)

**Restatement:** Under the given conditions, there is a point in the open interval where the value of the function is equal to the average value of the function over the interval.

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the value of the function corresponds to the height of a rectangle, with base  $(b-a)$ , whose area is the same as the area under the curve between the two endpoints.

91. Def. **Inverse function:** A function  $g$  is the inverse of the function  $f$  if  $f(g(x)) = x$  for each  $x$  in the domain of  $g$  and  $g(f(x)) = x$  for each  $x$  in the domain of  $f$ . Notation:

$$f^{-1}(x) = g(x).$$

92. Thm. **Basic Properties of Inverse Functions:**

- If  $(a, b)$  is a point on  $f(x)$ , then  $(b, a)$  is a point on  $f^{-1}(x)$ .
- The domain of  $f$  is the range of  $f^{-1}$  and the range of  $f$  is the domain of  $f^{-1}$ .
- The graphs of  $f$  and  $f^{-1}$  are mirror reflections about the diagonal line  $y = x$ .

93. Thm. **Existence of an Inverse:** A function possess an inverse if and only if it is one-to-one.

94. Def. **Strictly monotonic:** A function is strictly monotonic if it is either increasing on its entire domain or decreasing on its entire domain.

95. Thm. **Monotonic Implies 1:1:** If a function is strictly monotonic on an interval, then it is one-to-one on the interval and has an inverse.

96. Thm. **Advanced Properties of Inverse Functions:**

- If  $f$  is continuous, then  $f^{-1}$  is continuous.
- If  $f$  is increasing, then  $f^{-1}$  is increasing.
- If  $f$  is decreasing, then  $f^{-1}$  is decreasing.
- If  $f$  is differentiable at  $x = c$  and  $f'(c) \neq 0$ , then  $f^{-1}$  is differentiable at  $f(c)$ .

❖ 97. Thm. **Derivative of an Inverse Function:** If  $f$  is a differentiable function that possesses an inverse function  $g$ , then  $g'(x) = \frac{1}{f'(g(x))}$ .

**Restatement:** If  $f$  and  $g$  are inverse functions, and  $(a, b)$  is a point on the function  $g$ , then  $g'(a) = \frac{1}{f'(b)}$ .

98. Def. **Differential equation:** A differential equation in  $x$  and  $y$  is an equation that involves  $x$ ,  $y$ , and derivatives of  $y$ .

❖ 99. Def. **Solution of a differential equation:** A solution of a differential equation is a function or relation in  $x$  and  $y$  that is *defined on an open interval* and makes the differential equation true.