# 74 Definitions and Theorems You Need to Know

*All of the definitions and theorems below are required for the AP Calculus Examination. Particular attention should be given those marked with the ❖ symbol.*

1. Def. **Limit**: $\lim_{x\to c} f(x) = L$ means that *f(x)* becomes increasingly close to $L$ as $x$ gets closer to $c$.

❖ 2. Thm. **Limit Existence**: If *f(x)* is a function and $c$ and $L$ are real numbers, then $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

3. Thm. **Properties of Limits**: Let $b$ and $c$ be real numbers, let $n$ be a positive integer, and let $f$ and $g$ be functions whose limit at $c$ exists. Then
   a. Constant Function: $\lim_{x\to c} b = b$.
   b. Scalar Multiple: $\lim_{x\to c}\left[b \bullet f(x)\right] = b \bullet \lim_{x\to c} f(x)$.
   c. Sum or Difference: $\lim_{x\to c}\left[f(x) \pm g(x)\right] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$.
   d. Product: $\lim_{x\to c}\left[f(x) \bullet g(x)\right] = \lim_{x\to c} f(x) \bullet \lim_{x\to c} g(x)$.
   e. Quotient: $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$.
   f. Power: $\lim_{x\to c}\left[f(x)\right]^n = \left[\lim_{x\to c} f(x)\right]^n$.

4. Thm. **Squeeze Law**: If $h(x) \le f(x) \le g(x)$ for all $x$ in an open interval containing $c$, except possibly at $c$ itself, and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$, then $\lim_{x\to c} f(x) = L$.

5. Thm. **Trig Limit for the Sine**: $\lim_{x\to 0}\frac{\sin x}{x} = 1$ for $x$ measured in *radians*.

6. Thm. **Trig Limit for the Cosine**: $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$.

7. Def. **Horizontal asymptote**: A horizontal asymptote is a line $y = L$ such that $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$.

❖ 8. Def. **Continuity at a point**: A function *f(x)* is continuous at $c$ if $\lim_{x \to c} f(x) = f(c)$.

9. Def. **Continuity on an open interval**: A function is continuous on an open interval $(a, b)$ if it is continuous at each point in the interval.

10. Def. **Continuity on a closed interval**: A function is continuous on a closed interval $[a, b]$ if it is continuous on $(a, b)$, $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to b^-} f(x) = f(b)$.

11. Def. **Discontinuity**: A function $f$ is discontinuous at $c$ if $f$ is defined on an open interval containing $c$ (except possibly at $c$) and $f$ is not continuous at $c$.

12. Def. **Removable discontinuity**: A discontinuity at $x = c$ is called removable if $f$ can be made continuous by appropriately defining (or redefining) the single point $x = c$.

13. Thm. **Properties of Continuity**: If $b$ is a real number and *f(x)* and *g(x)* are continuous at $x = c$, then each of the following functions are also continuous at $c$:

   a. Scalar Multiple: $b \bullet f(x)$

   b. Sum or Difference: $f(x) \pm g(x)$

   c. Product: $f(x) \bullet g(x)$

   d. Quotient: $\dfrac{f(x)}{g(x)}$ provided $g(c) \neq 0$

14. Thm. **Continuity for Polynomial and Rational Functions**: Polynomial functions are everywhere continuous. Rational functions are continuous on their domain.

15. Thm. **Continuity for Composite Functions**: If *g(x)* is continuous at $c$ and *f(x)* is continuous at *g(c)*, then the composite function given by *f(g(x))* is continuous at $x = c$.

16. Def. **Average velocity**: The average velocity of an object over an interval of time is the net change in position during the interval divided by the change in time. For a function *s(t)*, that is $\bar{v} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$.

17. Def. **Instantaneous velocity**: The instantaneous velocity of an object at time $t_1$ is given by the limit of the average velocity as $t_2$ approaches $t_1$. For the function *s(t),* that is

$$v(t_1) = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

❖ 18. Def. **Difference quotient**: The expression $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called a difference quotient and represents the average rate of change of *f(x)* over the interval $[x_1, x_2]$.

Alternate notation for a difference quotient is $\dfrac{f(x+h) - f(x)}{h}$ or $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

❖ 19. Def. **Derivative**: $f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ (if it exists) is called the derivative of *f(x)* at $x_1$ and represents the instantaneous rate of change of *f(x)* at the point $x_1$.

❖ 20. Def. **Derivative, alternate forms**: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (if it exists) or $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ (if it exists).

21. Def. **Right-hand derivative**: The right-hand derivative of $f(x)$ is $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

22. Def. **Left-hand derivative**: The left-hand derivative of $f(x)$ is $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

23. Def. **Slope of a curve**: The slope of a curve at a point is the slope of the tangent line at the point.

❖ 24. Thm. **Differentiability and Continuity**: If a curve is differentiable at a point $x = c$, then it is continuous at $x = c$.

25. Def. **Local linearity**: A curve is called locally linear over an interval when zooming in on the curve causes it to look like a straight line.

26. Thm. **Local Linearity and Differentiability**: If a curve is locally linear at a point $x = c$ and the tangent line is not vertical there, then the function is differentiable at $x = c$.

27. Def. **Normal line**: A normal line to a curve at a point is a line perpendicular to the tangent line at the point.

❖ 28. Thm. **Properties of Derivatives**:

a. If $y = c$, then $y' = 0$.

b. If $y = c \bullet f(x)$, then $y' = c \bullet f'(x)$.

c. If $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$.

d. If $y = x^n$, then $y' = nx^{n-1}$.

29. Thm. **Product Rule**: If *f(x)* and *g(x)* are differentiable functions at *x*, then $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$.

30. Thm. **Quotient Rule**: If *f(x)* and *g(x)* are differentiable functions at *x*, and $g(x) \neq 0$, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$.

❖ 31. Thm. **Derivatives of the Trigonometric Functions**:

a. **Sine**: If $y = \sin x$, then $y' = \cos x$.

b. **Cosine**: If $y = \cos x$, then $y' = -\sin x$.

c. **Tangent**: If $y = \tan x$, then $y' = \sec^2 x$.

d. **Cotangent**: If $y = \cot x$, then $y' = -\csc^2 x$.

e. **Secant**: If $y = \sec x$, then $y' = \sec x \tan x$.

f. **Cosecant**: If $y = \csc x$, then $y' = -\csc x \cot x$.

❖ 32. Thm. **Chain Rule**: If $y = f(g(x))$ is a differentiable function of $g(x)$, and $g(x)$ is a differentiable function of $x$, then $\frac{d}{dx}[f(g(x))] = f'(g(x)) \bullet g'(x)$.

❖ 33. Thm. **Alternate Form of Chain Rule**: If $y = f(u)$ is a differentiable function of $u$, and $u = g(x)$ is a differentiable function of $x$, then $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$.

34. Thm. **Absolute Value Rule**: If $y = |x|$, then $y' = \frac{|x|}{x}$.

❖ 35. Thm. **Intermediate Value Theorem**: If *f(x)* is continuous on [*a, b*] and *k* is any number between *f(a)* and *f(b)*, then there is at least one number *c* between *a* and *b* such that $f(c) = k$.

*Geometric Application*: Under the given conditions, if *f(a)* and *f(b)* have opposite signs, then there is a point in the open interval where the graph crosses the *x*-axis.

❖ 36. Thm. **Extreme Value Theorem**: If $f(x)$ is continuous on [*a, b*], then *f* has both a maximum and minimum value on the interval.

❖ 37. Def. **Critical point**: If *f* is defined at *c,* and $f'(c) = 0$ or $f'(c)$ is undefined, then *c* is called a critical point of *f.*

38. Thm. **Finding Relative Extrema**: If *f* has a relative maximum or minimum at $x = c$, then *c* is a critical number of *f.*

❖ 39. Thm. **Rolle's Theorem**: If *f* **is** continuous on [*a, b*], differentiable on (*a, b*), and $f(a) = f(b)$, then there is at least one number *c* in (*a, b*) such that $f'(c) = 0$.

❖ 40. Thm. **Mean Value Theorem**: If *f* is continuous on [*a, b*] and differentiable on (*a, b*), then there exists a number *c* in (*a, b*) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the tangent to the curve is the same as the slope of the line joining the endpoints.

**Application:** Under the given conditions, there is a point in the open interval where the instantaneous rate of change is the same as the average rate of change on the interval. If the function is a position function, then there is a point in the open interval where the instantaneous velocity is the same as the average velocity on the interval.

41. Def. **Increasing function**: A function $f(x)$ is increasing on an interval if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

42. Def. **Decreasing function**: A function $f(x)$ is decreasing on an interval if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

❖ 43. Thm. **Interpreting the Derivative**: Let $f(x)$ be a function that is differentiable on the open interval $(a, b)$. Then:

a. If $f'(x) > 0$ for all $x$ in $(a, b)$, then $f(x)$ is increasing on $(a, b)$.

b. If $f'(x) < 0$ for all $x$ in $(a, b)$, then $f(x)$ is decreasing on $(a, b)$.

c. If $f'(x) = 0$ for all $x$ in $(a, b)$, then $f(x)$ is constant on $(a, b)$.

❖ 44. Thm. **First Derivative Test for Local Extrema**: Let $c$ be a critical point of the function $f$ that is continuous on an open interval. If $f$ is differentiable on the interval, except possible at $c$, then $f(c)$ can be classified as follows.

a. If $f'$ changes from negative to positive at $c$, then $f$ has a relative minimum at $c$.

b. If $f'$ changes from positive to negative at $c$, then $f$ has a relative maximum at $c$.

c. If $f'$ does not change signs at $c$, then $f$ has neither a relative maximum nor a relative minimum at $c$.

❖ 45. Def. **Concave upward**: The graph of a differentiable function $f$ is concave upward on an interval if $f'$ is increasing on the interval.

❖ 46. Def. **Concave downward**: The graph of a differentiable function $f$ is concave downward on an interval is $f'$ is decreasing on the interval.

❖ 47. Thm. **Test for Concavity**: Let $f(x)$ be a function whose second derivative exists on an open interval $(a, b)$. Then:

a. If $f''(x) > 0$ for all $x$ in $(a, b)$, then the graph of $f(x)$ is concave upward.

b. If $f''(x) < 0$ for all $x$ in $(a, b)$, then the graph of $f(x)$ is concave downward.

❖ 48. Def. **Inflection point**: If the graph of f changes concavity at c and there exists a tangent line to the curve at c, then c is called an inflection point of *f*.

❖ 49. Thm. **Finding Inflection Points with the Second Derivative**: If $c$ is a point of inflection of the graph of *f*, then either $f''(c) = 0$ for $f''$ is undefined at $x = c$.

❖ 50. Thm. **Second Derivative Test for Relative Extrema**: Let $f$ be a function such that $f'(c)=0$ and the second derivative of $f$ exists on some open interval containing $c$.

a. If $f''(c)>0$, then $f(c)$ is a relative minimum.

b. If $f''(c)<0$, then $f(c)$ is relative maximum.

c. If $f''(c)=0$ or does not exist, the test fails.

❖ 51. Def. **Differential of *y***: Let $y=f(x)$ represent a differentiable function and let $dx$ be any nonzero change in $x$. Then the differential of $y$, written $dy$ is given by $dy=f'(x)dx$.

52. Def. **Linear approximation of $\Delta y$**: $dy$ is called the linear approximation of the actual increment, $\Delta y$.

❖ 53. Def. **Linear approximation of *f(x)***: The expression $f(x)\approx f'(a)(x-a)+f(a)$ is called the linear approximation to *f(x)* near $x=a$.

❖ 54. Def. **Riemann sum**: Let $f$ be defined on the closed interval $[a, b]$ which is partitioned by the set $\{a=x_0, x_1, x_2, \cdots, x_n=b\}$. If $c_i\in[x_{i-1}, x_i]$ and $\Delta x_i=x_i-x_{i-1}$, then the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann sum of $f$ for the given partition.

❖ 55. Def. **Definite integral**: If $f$ is defined on the interval $[a, b]$ and the limit of the Riemann sum $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then this limit is called the definite integral of $f$ on $[a, b]$ and is denoted by $\int_a^b f(x)dx=\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i$. (The values $a$ and $b$ are called the lower and upper limits of the integral, respectively.)

56. Def. **Existence of the definite integral**: If $f$ is a continuous function on $[a, b]$, then the limits as $n\to\infty$ of both the lower and upper sums exist and are equal to each other.

❖ 57. Thm. **Continuity and Integrability**: If a function $f$ is continuous on the closed interval $[a, b]$, then $f$ is integrable on $[a, b]$.

58. Thm. **Area Under a Curve**: Let $f$ be a continuous, function on the interval $[a, b]$. The area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x=a$ and $x=b$ is found by:

$$area=\int_a^b |f(x)|dx.$$

59. Thm. **Integration of an Odd Function**: If $f$ is an odd function which is integrable on $[-a, a]$, then $\int_{-a}^{a} f(x)dx=0$.

60. Thm. **Integration of an Even Function**: If $f$ is an even function which is integrable on [-*a*, *a*], then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

61. Thm. **Properties of the Definite Integral:** Given that $f$ and $g$ are two integrable functions on [*a*, *b*].

   a. **At a point**: $\int_{a}^{a} f(x)dx = 0$.

   b. **Over a reverse interval**: $\int_{b}^{a} f(x)dx = -\int_{a}^{b} f(x)dx$.

   c. **Two Adjacent Intervals**: $\int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx = \int_{a}^{b} f(x)dx$

   d. **Constant Times a Function**: $\int_{a}^{b} k \bullet f(x)dx = k \bullet \int_{a}^{b} f(x)dx$.

   e. **Sum or Difference of Functions**: $\int_{a}^{b} [f(x) \pm g(x)]dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$.

62. Thm. **Comparing Definite Integrals**: If $f$ and $g$ are both integrable functions on [*a*, *b*] and $f(x) \le g(x)$ for $a \le x \le b$, then $\int_{a}^{b} f(x)dx \le \int_{a}^{b} g(x)dx$.

63. Def. **Antiderivative**: If *f′(x)* is the derivative of *f(x)*, then *f(x)* is called an antiderivative of *f′(x)*.

64. Thm. **Relationship Between Antiderivatives**: If $F$ and $G$ are both antiderivatives of a function *f*, then $F(x) = G(x) + C$ for some constant *C*.

65. Thm. **Distance versus Displacement**: If the continuous function *v(t)* represents the velocity of a function over an interval of time [*a*, *b*], then:

   a. Displacement (or change in position) = $\int_{a}^{b} v(t)dt$.

   b. Total Distance Traveled = $\int_{a}^{b} |v(t)|dt$

❖ 66. Thm. **Fundamental Theorem of Calculus**: If a function $f$ is continuous on the closed interval [*a*, *b*] and $F$ is an antiderivative of $f$ on the interval [*a*, *b*], then $\int_{a}^{b} f(x)dx = F(b) - F(a)$.

**Restatement**: The total change in a function is the definite integral of its rate of change.

❖ 67. Def. **Average value of a function**: If $f$ is a continuous function on $[a, b]$, then the average value of $f$ on $[a, b]$ is found by $\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$.

❖ 68. Thm. **Average Value Theorem for Integrals**: If $f(x)$ is continuous on $[a, b]$, then there exists a number $c$ in $(a, b)$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. (Note that in the Larson text this theorem is referred to as the Mean Value Theorem for Integrals.)

**Restatement:** Under the given conditions, there is a point in the open interval where the value of the function is equal to the average value of the function over the interval.

**Geometric Interpretation:** Under the given conditions, there is a point in the open interval where the value of the function corresponds to the height of a rectangle, with base $(b-a)$, whose area is the same as the area under the curve between the two endpoints.

69. Def. **Accumulation function**: Let $f$ be continuous on $[a, b]$ and $x \in [a,b]$. The function $A(x) = \int_a^x f(t)dt$ is called the accumulation function.

❖ 70. Thm. **Second Fundamental Theorem of Calculus**: If $f$ is continuous on an open interval containing $a$, then for every $x$ in the interval where $F(x) = \int_a^x f(t)dt$, *F(x)* is an antiderivative of *f(x)*.

71. Def. **Indefinite integral**: The expression $\int f(x)dx$ is called the indefinite integral and is used to represent the family of all antiderivatives of $f(x)$.

72. Thm. **Properties of the Indefinite Integral**:

a. $\int dx = x + C$.

b. **Constant Times a Function**: $\int k \bullet f(x)dx = k \bullet \int f(x)dx$.

c. **Sum or Difference of Functions**: $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$.

❖ 73. Thm. **Pythagorean Trigonometric Properties**:

a. $\sin^2 x + \cos^2 x = 1$

b. $\tan^2 x + 1 = \sec^2 x$

c. $\cot^2 x + 1 = \csc^2 x$

❖ 74. Thm. **Integrals of the Trigonometric Functions**:

a. **Sine**: $\int \sin x dx = -\cos x + C$ .

b. **Cosine**: $\int \cos x dx = \sin x + C$ .

c. **Tangent**: $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$ .

d. **Cotangent**: $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$ .

e. **Secant**: $\int \sec x dx = \ln|\sec x + \tan x| + C$ .

f. **Cosecant**: $\int \csc x dx = -\ln|\csc x + \cot x| + C$ .