TRANSCENDENTAL FUNCTIONS

1. Def. **e**: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ or $e = \lim_{n\to 0}(1+n)^{\frac{1}{n}}$.

2. Def. **Exponential Function**: $y = a^x$ where $a > 0$ and $a \neq 1$ is called the exponential function.

3. Def. **Logarithm Function**: If $y = \log_a x$, then $a^y = x$, where $a > 0$, $a \neq 1$, and $x > 0$. Notation: $\ln x = \log_e x$ and $\log x = \log_{10} x$.

4. Thm. **Properties of Logarithms**:

   a. $\log_b 1 = 0$.

   b. $\log_b pq = \log_b p + \log_b q$.

   c. $\log_b \frac{p}{q} = \log_b p - \log_b q$.

   d. $\log_b p^n = n \log_b p$.

   e. $\log_a x = \frac{\log_b x}{\log_b a}$.

   f. $b^{\log_b x} = x$.

   g. $\log_b b^x = x$.

5. Thm. **Properties of the Natural Logarithm.**

   a. The domain of $y = \ln x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.

   b. The function is continuous, increasing, one-to-one.

   c. The graph is concave downward on its entire domain.

   d. $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$.

6. Thm. **Derivative of the Natural Logarithm.** If $y = \ln x$, then $y' = \frac{1}{x}$.

7. Thm. **Derivative of the Natural Logarithm with Absolute Value**: If $y = \ln|x|$, then $y' = \frac{1}{x}$.

8. Thm. **Integration of $\frac{1}{x}$**: $\int \frac{1}{x} dx = \ln|x| + C$ for $x \neq 0$.

9. Thm. **Integral of the Sine Function**: $\int \sin x dx = -\cos x + C$.

10. Thm. **Integral of the Cosine Function**: $\int \cos x dx = \sin x + C$.

11. Thm. **Integral of the Tangent Function**: $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$.

12. Thm. **Integral of the Cotangent Function**: $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$.

13. Thm. **Integral of the Secant Function**: $\int \sec x dx = \ln|\sec x + \tan x| + C$.

14. Thm. **Integral of the Cosecant Function**: $\int \csc x dx = -\ln|\csc x + \cot x| + C$.

--*End for Chapter 5 Quiz*--

15. Def. **Inverse Function**: A function g is the inverse of the function f if $f(g(x)) = x$ for each x in the domain of g and $g(f(x)) = x$ for each x in the domain of *f*. Notation: $f^{-1}(x) = g(x)$.

16. Thm. **Basic Properties of Inverse Functions**:

    a. If (a, b) is a point on $f(x)$, then (b, a) is a point on $f^{-1}(x)$.

    b. The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

    c. The graphs of f and f^{-1} are mirror reflections about the diagonal line $y = x$.

17. Def. **One-to-One**: A function is called one to one if no two ordered pairs have the same first or second member.

18. Thm. **Existence of an Inverse**: A function possess an inverse if and only if it is one-to-one.

19. Def. **Strictly Monotonic**: A function is strictly monotonic if it is either increasing on its entire domain or decreasing on its entire domain.

20. Thm. **Monotonic Implies 1:1**: If a function is strictly monotonic on an interval, then it is one-to-one on the interval and has an inverse.

21. Thm. **Advanced Properties of Inverse Functions**:

a. If f is continuous, then f^{-1} is continuous.

b. If f is increasing, then f^{-1} is increasing.

c. If f is decreasing, then f^{-1} is decreasing.

d. If f is differentiable at $x = c$ and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

22. Thm. **Derivative of an Inverse Function**: If f is a differentiable function that possesses an inverse function g, then $g'(x) = \dfrac{1}{f'(g(x))}$.

Restatement: If f and g are inverse functions, and (a, b) is a point on the function g, then $g'(a) = \dfrac{1}{f'(b)}$.

23. Thm. **Properties of the Natural Exponential Function:**

a. The domain of $y = e^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$.

b. The function is continuous, increasing, one-to-one.

c. The graph is concave upward on its entire domain.

d. $\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to +\infty} e^x = \infty$.

24. Thm. **Derivative of the Natural Exponential Function**: If $y = e^x$, then $y' = e^x$.

25. Thm. **Integration of e^x**: $\int e^x \, dx = e^x + C$.

26. Thm. **Derivative of an Exponential Function**: If $y = a^x$, then $y' = a^x \ln a$.

27. Thm. **Integration of a^x**: $\int a^x \, dx = \dfrac{1}{\ln a} a^x + C$.

28. Thm. **Derivative of a Logarithm.** If $y = \log_a x$, then $y' = \dfrac{1}{x \ln a}$.

29. Thm. **Continuous Compounding**: If an amount A_0 is invested at interest rate r compounded continuously (or instantaneously), then the amount A accrued in time t is $A = Pe^{rt}$.

30. Def. **Differential Equation**: A differential equation in x and y is an equation that involves x, y, and derivatives of y.

31. Def. **Solution of a Differential Equation**: A solution of a differential equation is a function or relation in x and y that is *defined on an open interval* and makes the differential equation true.

❖ 33. Thm. **Euler's Method**: For a differentiable relation in y and x where y' is known and $y(x_0)$ is known, then any value of y can be approximated by the iterative procedure $y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$.

34. Def. **Stable Equilibrium**: A relationship where the dependent variable, regardless of its initial value, tends toward a constant state as the independent variable tends to infinity, is called a stable equilibrium.

35. Def. **Equilibrium Solution**: The constant value to which the dependent variable approaches in a stable equilibrium is called the equilibrium solution.

36. Thm. **Stable Equilibriums**: Differential equations of the form $\frac{dy}{dt} = -k(y - E)$, where $k > 0$, are stable equilibriums.

37. Thm. **Equilibrium Solution**: The value for y which makes the derivative zero is the equilibrium solution.

38. Thm. **Newton's Law of Cooling**: The rate at which a body cools is proportional to the difference between its temperature and that of the surrounding air.

39. Thm. **Removal of Pollutants**: (You are not required to learn this theorem.) If a lake of volume V with water flow r contains an amount Q of pollutants, then the rate of change of the pollutants with respect to time t can be described as $\frac{dQ}{dt} = \frac{-r}{V}Q$, presuming no additional pollutants enter the lake.

40. Def. **Cooling Curve**: A cooling curve is a curve whose differential equation is of the form $\frac{dy}{dt} = -k(y - E)$, where k is positive and $y_0 > E$.

41. Thm. **Characteristics of the Cooling Curve**: The cooling curve:

   a. Has a growth rate that is always negative.

   b. Decreases rapidly at the start and more slowly as time passes.

   c. Is a stable equilibrium, with equilibrium solution $y = E$.

42. Def. **Learning Curve**: A learning curve is a curve whose differential equation is of the form $\frac{dy}{dt} = -k(y - E)$, where k is positive and $y_0 < E$.

43. Thm. **Characteristics of the Learning Curve**: The learning curve:

    a. Has a rate of growth that is always positive.

    b. Increases rapidly at the start and more slowly as time passes.

    c. Is a stable equilibrium, with equilibrium solution $y = E$.

--*End for Chapter 5 Test*--

44. Def. **Inverse Sine**: $y = \arcsin x$ means $x = \sin y$ where $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

45. Def. **Inverse Cosine**: $y = \arccos x$ means $x = \cos y$ where $0 \le y \le \pi$.

46. Def. **Inverse Tangent**: $y = \arctan x$ means $x = \tan y$ where $\frac{-\pi}{2} < y < \frac{\pi}{2}$.

47. Def. **Inverse Cotangent**: $y = \operatorname{arccot} x$ means $x = \cot y$ where $0 < y < \pi$.

48. Def. **Inverse Secant**: $y = \operatorname{arcsec} x$ means $x = \sec y$ where $0 \le y \le \pi$ and $y \ne \frac{\pi}{2}$.

49. Def. **Inverse Cosecant**: $y = \operatorname{arccsc} x$ means $x = \csc y$ where $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ and $y \ne 0$.

50. Thm. **Derivative of the Inverse Sine**: If $y = \arcsin x$, then $y' = \frac{1}{\sqrt{1-x^2}}$.

51. Thm. **Derivative of the Inverse Cosine**: If $y = \arccos x$, then $y' \frac{-1}{\sqrt{1-x^2}}$.

52. Thm. **Derivative of the Inverse Tangent**: If $y = \arctan x$, then $y' = \frac{1}{1+x^2}$.

53. Thm. **Derivative of the Inverse Cotangent**: If $y = \operatorname{arccot} x$, then $y' = \frac{-1}{1+x^2}$.

54. Thm. **Derivative of the Inverse Secant**: If $y = \operatorname{arcsec} x$, then $y' = \frac{1}{|x|\sqrt{x^2-1}}$.

55. Thm. **Derivative of the Inverse Cosecant**: If $y = \operatorname{arccsc} x$, then $y' = \frac{-1}{|x|\sqrt{x^2-1}}$.

56. Thm. **Integration with the Inverse Sine Function**: $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$.

57. Thm. **Integration with the Inverse Tangent Function**: $\int \frac{1}{1+x^2}dx = \arctan x + C$.

58. Thm. **Integration with the Inverse Secant Function**: $\int \frac{1}{x\sqrt{x^2-1}} = arc\sec|x| + C$.

❖ 59. Thm. **L'Hôpital's Rule**: If $\lim\left[\frac{f(x)}{g(x)}\right]$ results in the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$\lim\left[\frac{f(x)}{g(x)}\right] = \lim\left[\frac{f'(x)}{g'(x)}\right]$ provided the latter limit is of a determinate form

---*End for Chapter 6 Quiz*---