DERIVATIVE OF THE LOGARITHM

Thm. **Derivative of the Natural Logarithm**: If $y = \ln x$, then $y' = \frac{1}{x}$.

Proof:

$$f'(x) = \lim_{h \to 0} \frac{\qquad\qquad}{} \qquad \text{Definition of the derivative}$$

$$f'(x) = \lim_{h \to 0} \frac{\ln\left(\qquad \right)}{h} \qquad \text{Property of Logarithms}$$

$$f'(x) == \lim_{h \to 0} \left[\frac{1}{x} \frac{x}{h} \ln\left(\frac{x+h}{x} \right) \right] \qquad \text{Properties of Fractions}$$

$$f'(x) = \lim_{h \to 0} \left[\frac{1}{x} \ln\left(1 + \frac{h}{x} \right)^{\frac{x}{h}} \right] \qquad \underline{\qquad\qquad\qquad}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{x} \bullet \lim_{h \to 0} \left[\ln\left(1 + \frac{h}{x} \right)^{\frac{x}{h}} \right] \qquad \underline{\qquad\qquad\qquad}$$

$$f'(x) = \frac{1}{x} \bullet \lim_{h \to 0} \left[\ln\left(1 + \frac{h}{x} \right)^{\frac{x}{h}} \right] \qquad \underline{\qquad\qquad\qquad}$$

$$f'(x) = \frac{1}{x} \bullet \ln\left[\lim_{h \to 0} \left(1 + \frac{h}{x} \right)^{\frac{x}{h}} \right] \qquad \underline{\qquad\qquad\qquad}$$

$$f'(x) = \frac{1}{x} \bullet \ln(\qquad) \qquad \underline{\qquad\qquad\qquad}$$

$$f'(x) = \qquad \underline{\qquad\qquad\qquad}$$

DERIVATIVE OF THE LOGARITHM AND APPLICATIONS

Thm. **Derivative of a Logarithm**: If $y = \log_a x$, then $y' = \dfrac{1}{x \ln a}$.

Proof: Use a change of base formula to write y in terms of ln. Then differentiate.

Thm. **Continuous Compounding**: If an amount A_0 is invested at interest rate r compounded continuously, then the amount A accrued in time t is $A = A_0 e^{rt}$.

1. How does the continuous rate of interest compare to the effective annual rate of interest?

   a. The effective annual rate of interest is just simple interest. Let's call it R.

   Calculate the simple interest for one year using the formula $I = PRT$, where we represent P as A_0.

   Calculate the total amount accumulated by the end of one year.

   Solve this equation for R.

   Since $A - A_o$ is ΔA and $\Delta T = 1$, we can write $R = \dfrac{\frac{\Delta A}{\Delta t}}{A_0}$.

   Hence, the rate of *simple interest* is a *difference quotient* over the initial amount. It corresponds to an *average rate of change* over the time interval of one year.

b. Let r be the continuous rate of interest referred to in the formula $A = A_o e^{rt}$.

Consider the derivative of the formula $A = A_0 e^{rt}$.

$$\frac{dA}{dt} = .$$

But substituting the first equation into the second we get

$$\frac{dA}{dt} =$$

or $r = .$

The *continuous* rate of interest is the *derivative* over the current amount. It represents an *instantaneous* rate of change at any point in time during the year.

c. The above expressions can be generalized for any quantity that is growing at a continuous or instantaneous rate r.

2. The population, P, in millions, of Nicaragua was 3.6 million in 1990 and growing exponentially, with an effective annual growth rate of 3.4%. Find the continuous rate of growth.

In terms of the effective annual growth rate $R = 0.034$, we would say the population at the end of one year is

$P = P_0 +$ ______________

In terms of a continuous rate of growth r, we would say the population at the end of one year is

$P = P_0 \bullet$ ______________

Set these two expressions equal to each other and solve for r.

DIFFERENTIAL EQUATIONS AND SLOPE FIELDS

Def. **Differential Equation**: A differential equation in x and y is an equation that involves x, y, and derivatives of y.

Def. **Solution of a Differential Equation**: A solution of a differential equation is a function or relation in x and y that is *defined on an open interval* and makes the differential equation true.

3. Solve the differential equation: $\frac{dy}{dx} = 3x^2 + 4x$.

Recall that the derivative of a function tells us the slope of the line tangent to the curve at a point. Since curves are locally linear, this tangent line is a reasonable approximation to the original function near the point.

4. Construct a slope field for the differential equation: $\frac{dy}{dx} = 2x$ and draw the specific solution where $f(-3) = 2$

   a. Pick any point on a coordinate grid and use the coordinates of the point to calculate the derivative at that point.

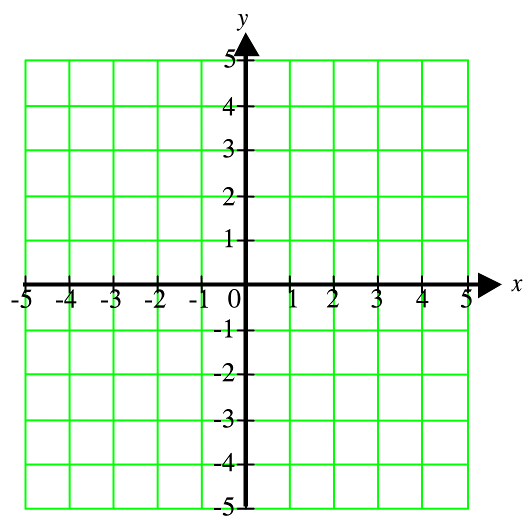
   b. Draw a short tangent line segment on the coordinate system at the chosen point.

   c. Repeat for multiple points on the coordinate system.

   d. Try to visualize the function from the slope field. Notice that many curves can be drawn depending on where one starts. The starting point is what we call the initial condition.

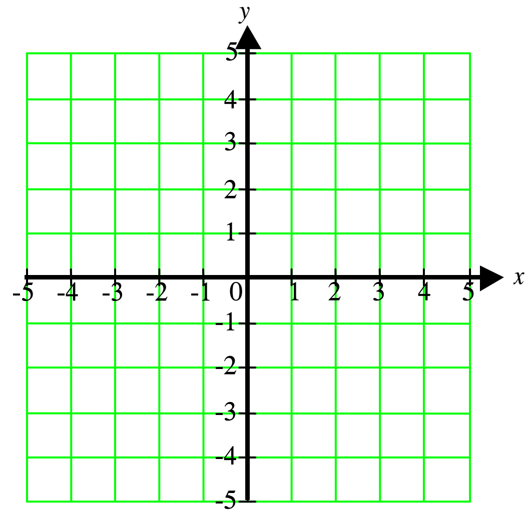
   e. Draw a specific solution where $f(-3) = 2$.

x	y	dy/dx



5. Construct a slope field for the differential equation: $\frac{dy}{dx} = \frac{1}{x}$ and draw several solutions.

x	*y*	*dy/dx*



6. Use the TI-89 to draw a slope field for $\frac{dy}{dx} = \frac{1}{x}$ and sketch a specific solution with $f(0.05) = -3$.

First set *MODE/Graph/Diff Equations*.

Set the window size to $0 \le x \le 2$, $-3 \le y \le 3$ and, fldres=16.

Open the Y= window and enter y_1' as $\frac{1}{t}$. (THE INDEPENDENT VARIABLE MUST BE *t*.) Let $t0 = 0$ and $yi1 =$ be blank.

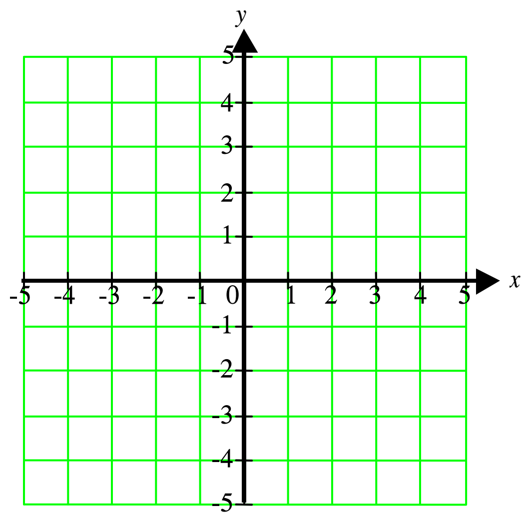
While in the Y= window, press *F1* *Format.* Check that the following are set:
Coord/RECT *Grid/OFF* *Solution Method/Euler* *Fields/SLPFLD*

GRAPH.

Open the Y= window. Let $t0 = 0.05$ and $yi1 = -3$. *GRAPH*.

7. Construct a slope field for the differential equation: $\frac{dy}{dx} = 2 + y$ and sketch the specific solution where $f(-2) = -1.5$.

x	*y*	*dy/dx*



8. Use the TI-89 to draw a slope field for $\frac{dy}{dx} = 2 + y$ and draw a specific solution with $f(-2) = -1.5$.

Open the Y= window and enter $y1 = 2 + y1$ (where y1 is entered using the y followed by the 1 off the keyboard)

Set window size to $-3 \le x \le 3$ and $-4 \le y \le 3$. Again use fldres=16. Let $t0 = 0$ and $yi1$ be blank

GRAPH

Go to the Y= window and make $t0 = -2$ and $yi1 = -1.5$

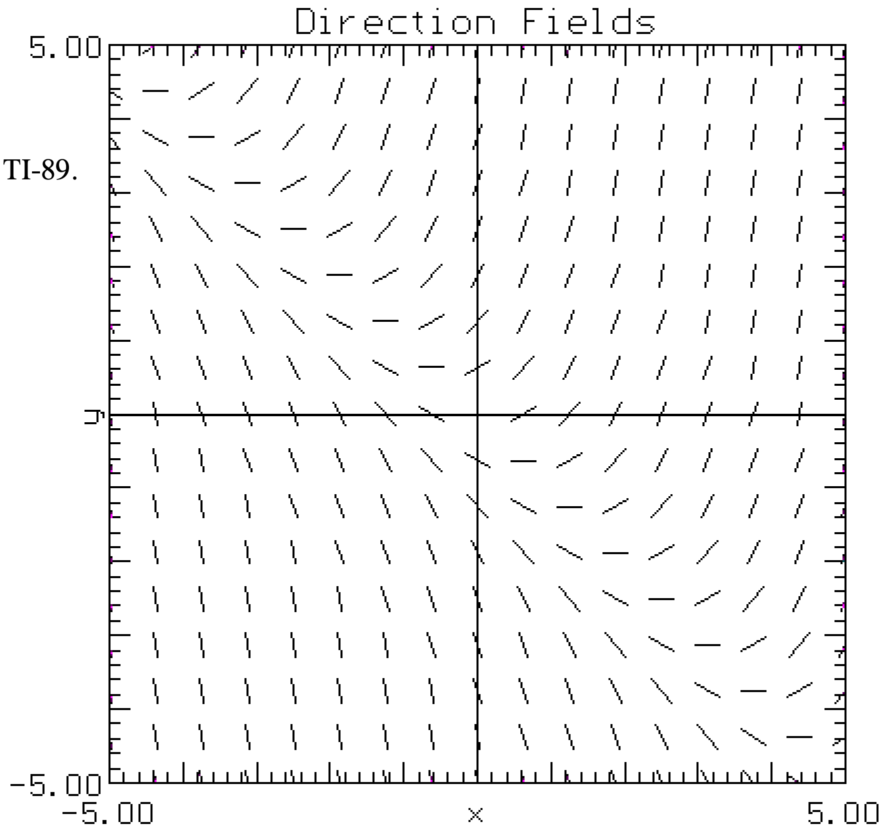
GRAPH

9. Use the slope field below to draw a solution to the differential equation $\frac{dy}{dx} = x + y$

   a. Where $f(-2) = 1.5$.

   b. Where $f(-3) = 0$.

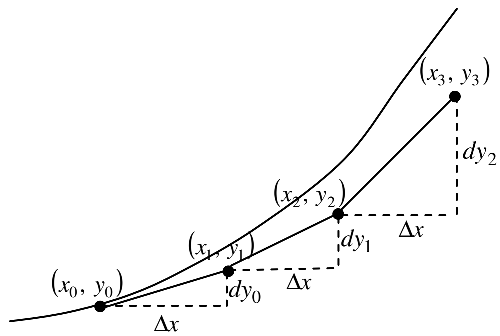
   c. Check both solutions on the TI-89.



EULER'S METHOD

In the previous lesson we saw how to sketch a solution curve to a differential equation using its direction field (or slopefield), whose line segments are everywhere tangent to the solution. In this lesson we will do the same thing numerically, but computing points on the solution curves using Euler's Method.

Here is the concept behind Euler's Method. Think of the direction field (or slopefield) as a set of signposts directing us across the plane. Pick a starting point (corresponding to the initial value), and calculate the slope at that point using the differential equation. This slope is a signpost telling us to the direction to take. Head off along a straight line path for a short distance in that direction. Stop and look at the new signpost. Recalculate the slope from the differential equation, using the coordinates of the new point. Change direction to correspond to the new slope, and move another small distance, and so on.



This can be summarized as:

(new y value) = (the old y value) + (slope of the curve at the old point) * (change in x).

Let $y'(x_i)$ be the slope of the function at the point (x_i, y_i). Then Euler's Formula is $y_{i+1} = y_i + y'(x_i) \bullet \Delta x_i$. Remember that these values are always approximations to the actual curve. The smaller the value of Δx, the better the approximation.

10. Many populations grow at a rate which is directly proportional to the population itself. Assume this pattern of growth for the world population. In 2005, the constant of proportionality was 0.013, with a world population estimate of 6.396 billion. What would you project the population to be in the year 2020?

    a. Write an expression for the derivative.

    b. Write an expression for the differential.

c. Compute dy over a fifteen year interval.

d. Why would we expect this value to be significantly off?

e. Compute dy over a *one* year interval, from 2005 to 2006.

f. Compute dy over a one year interval, from 2006 to 2007.

g. Use TI-89 to calculate the recursive pattern.
In the home screen type 6.396
ENTER
ANS + 0.013 • *Ans* • 1 *ENTER*
ENTER
ENTER
Continue in this manner until you get a value for the year 2020.

h. Redo with a 6-month time interval.

i. Solve analytically and compare answers.

11. The differential equation for interest which is compounded continuously is $\frac{dy}{dx} = ky$, where k is the rate of interest and y is the amount in the account. Suppose you have \$1000 invested at 6% interest compounded continuously. Use Euler's Method and your TI-89 to approximate the total amount you would have in 4 years, using intervals of 6 months.

EXPONENTIAL GROWTH AND DECAY

12. In a favorable environment, the number of bacteria increases at a rate proportional to the number present. If 1000 bacteria are present at a certain time and 2000 are present an hour later, find the number present four hours later.

13. A bank account grows continuously at a rate of 4.5%. If \$3000 is deposited in the account, how much will be in the account 6 years later? What is the effective annual interest rate?

14. A radioactive substance decays at a rate proportional to the amount present. If one gram of radioactive substance reduces to $\frac{1}{4}$ gm. in four hours, find:

    a. The half-life of the substance, and

    b. How long it will take until only $\frac{1}{10}$ gm. remains.

LEARNING CURVES AND COOLING CURVES

15. A body originally at 120° Fahrenheit cools to 100° in 10 minutes in air at 60°. Find the temperature of the body after it has cooled for 1 hour.

16. Suppose a lake of volume 2700 km^3 has an outflow of 90 km^3/yr. How long will it take for 60% of its pollutants to be removed?