APPROXIMATING DISTANCE

Suppose we are driving down the highway at varying speeds and we need to calculate the distance we will travel over the next hour. We can see the speedometer and we can see a clock. How can we get a reasonable approximation?

1. Use the table of data below to calculate an approximation. (t is in hours; v is in mph)

t	0	1/6	2/6	3/6	4/6	5/6	1
$v(t)$	25	37	45	58	61	64	66

2. Is the above approximation likely to be too small or too big?

3. Now calculate the "other sum".

4. What is the difference between our two approximations?

5. How can we shrink the difference between the two approximations?

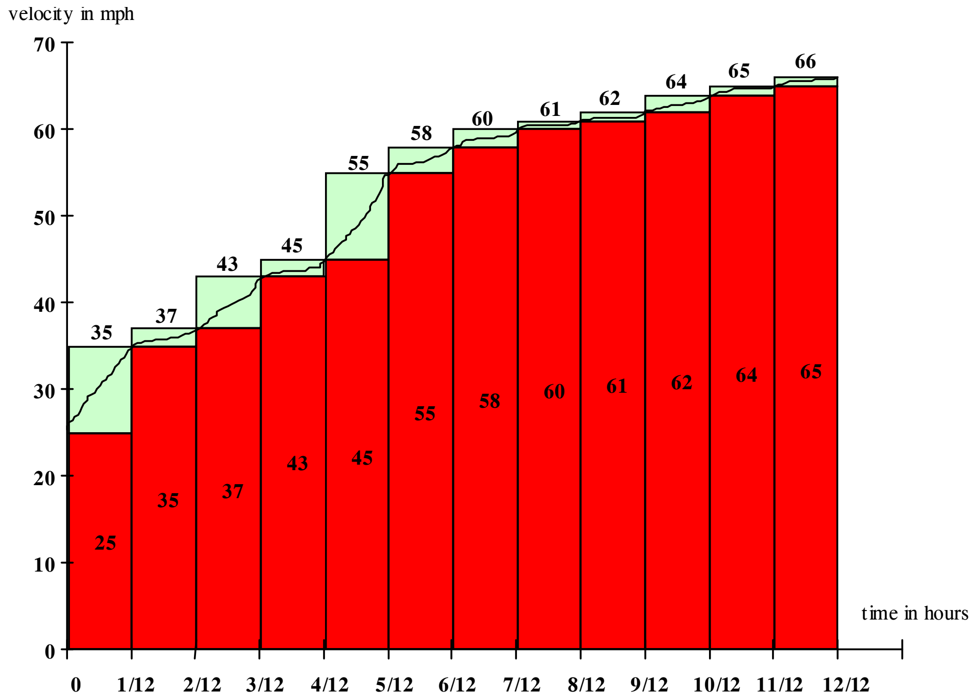
6. Use the table below to get another pair of lower and upper estimates.

t	0	1/12	2/12	3/12	4/12	5/12	6/12	7/12	8/12	9/12	10/12	11/12	1
$v(t)$	25	35	37	43	45	55	58	60	61	62	64	65	66

7. What is the difference between the two approximations this time?

8. How is the difference between these two approximations related to the difference between the first two approximations?

9. Consider a graph of the velocity function.



10. To what does the darker area correspond?

11. To what does the total shaded area correspond?

12. To what does the lighter shaded area alone correspond?

13. Find the measure of the lighter area visually.

Notice that if we took the lighter area and shoved it all into one vertical column, its width would be Δt and its height would be $v_{final} - v_{initial}$ for a *strictly increasing* (or strictly decreasing) function.

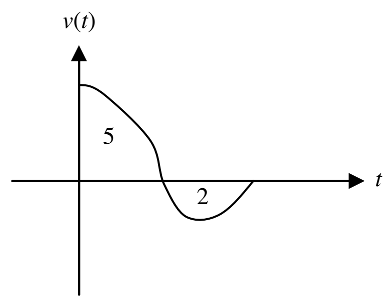
14. Can you now explain why halving the time interval above halved the difference between the upper and lower approximations for the distance traveled?

FINDING TOTAL CHANGE

We have already learned that distance traveled corresponds to the area under a non-negative velocity curve. Since the definite integral represents the area under a curve for a non-negative function, we can write that the distance traveled by a moving object over the time interval $[a, b]$ is $\int_a^b v(t)dt$, provided the velocity function is always positive.

Now suppose that the velocity function is sometimes positive and sometimes negative. To find the actual distance traveled, we need to find the definite integral of the *absolute value* of the velocity function, or $\int_a^b |v(t)|dt$.

What then does $\int_a^b v(t)dt$ represent? Consider the graph at right that has 5 units of area above the axis and 2 units of area below the axis. The area above the axis would be the distance moved forward, since the velocity curve is positive there. The area below the axis would be the distance moved backward, since the velocity curve is negative there. The difference in these two areas should be the *change in position* between the object's starting and ending points. Since the definite integral assigns positive values to area above the axis and negative values to area below the axis, the definite integral calculates this change in position.



If we let $s(t)$ be the position function, then $s'(t) = v(t)$. We can then express the change in position as: $s(b) - s(a) = \int_a^b s'(t)dt$.

The Fundamental Theorem of Calculus is a generalization of this concept. For any function $f(t)$, whose derivative is continuous, the change in value of $f(t)$ over the interval $[a, b]$ can be found by $f(b) - f(a) = \int_a^b f'(t)dt$. In summary, the total change in a function is the definite integral of its rate of change.

15. A company purchases a new machine for which the rate of depreciation is $V'(t) = 10{,}000(t-6)$, where V is the value of the machine after t years have elapsed. Find the total loss in value over the first 3 years. (Note: The rate of depreciation is the rate of change of the value.)

16. A population of bacteria is changing at the rate of $\frac{dP}{dt} = \frac{3000}{1+0.25t}$. where t is the time in days. The initial population (when $t = 0$) is 1000. Find the population in 3 days.

17. Find the *total distance* traveled by an object whose velocity is described as $v(t) = 3t^2 - 3$ over the interval [0, 4].

18. An object moves along the number line with velocity that can be described as $v(t) = t^3 - 6t + 2$. At time $t = 4$ it is at the number 32. Where was it at time $t = 0$?

RELATIONSHIP AMONG ANTIDERIVATIVES

Thm. **Relationship Between Antiderivatives**: If F and G are both antiderivatives of a function f, then $F(x) = G(x) + C$ for some constant C.

Proof: Let F and G be antiderivatives of f.

Construct the function $H(x) = F(x) - G(x)$.

If the theorem is true, then $H(x)$ will be constant.

Assume $H(x)$ is not constant. Then there exist two points a and b where $H(a) \neq H(b)$.

By the Mean Value Theorem there exists a point $c \in (a,b)$ such that:

$H'(c) =$

Since $H(a) \neq H(b)$, then $H(b) - H(a) \neq 0$.

Why can we conclude that $H'(c) \neq 0$?

However $H(x) = F(x) - G(x)$.

So, differentiating $H(x)$ gives us

$H'(x) =$

We have found that $H'(x) = 0$ for all x.

What does this statement contradict?

Hence our assumption is false and $H(x)$ must be constant.

AVERAGE VALUE OF A FUNCTION

19. What is meant by the average value of a function?

20. How would we find the average value of an infinite set of numbers?

Let $f(x)$ be defined over the interval $[a, b]$. Divide the interval into n subintervals of width Δx. An approximate average value would be

$\bar{y} \approx$

Express this result in summation notation:

$\bar{y} \approx$

Multiply the numerator and denominator by Δx.

$\bar{y} \approx$

What is $\Delta x * n$ equivalent to? Substitute this into the expression.

$\bar{y} \approx$

This is just an approximation for $\bar{y}$. How would we improve our approximation?

How could we get an exact value?

$\bar{y} =$

The numerator is equivalent to what expression?

$\bar{y} =$

Def. **Average Value of a Function**: If f is a continuous function on $[a, b]$, then the average value of f on $[a, b]$ is found by $\bar{y} = \frac{1}{b-a}\int_a^b f(x)dx$.

21. Find the average value of the function $y = 6x^2 - 7x$ on the interval $[-1, 1]$.

22. The volume V in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model $V = 0.1729t + 0.1522t^2 - 0.0374t^3$ where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle. (Use TI-89)

Thm. **Average Value Theorem for Integrals**: If $f(x)$ is continuous on $[a, b]$, then there exists a number c in (a, b) such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. (Note that in the Larson text this theorem is referred to as the Mean Value Thm for Integrals.)

Numerical implications of the theorem:

Graphical implications of the theorem:

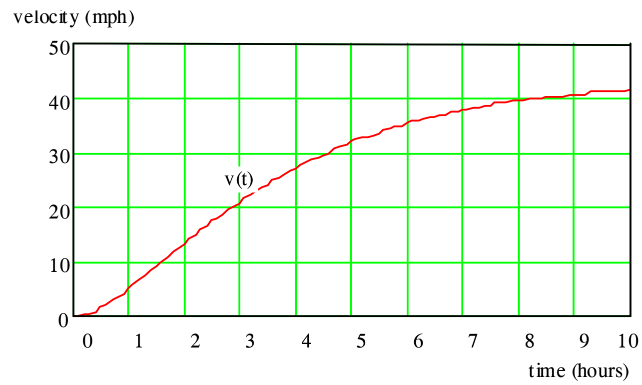
Analytic implications of the theorem:

23. Find the value of c guaranteed by the Average Value Theorem for Integrals for the function $y = \sin(x)$ on the interval $\left[0, \frac{\pi}{2}\right]$

CONSTRUCTING THE DISTANCE FUNCTION

24. Suppose we have a car whose velocity is graphed at right. We want to construct a function to represent the distance traveled at any time x.

a. Let $s(x)$ be the function we are defining. Estimate values for the function at each integer point.



s(0)≈ s(5)≈

s(1)≈ s(6)≈

s(2)≈ s(7)≈

s(3)≈ s(8)≈

s(4)≈ s(9)≈

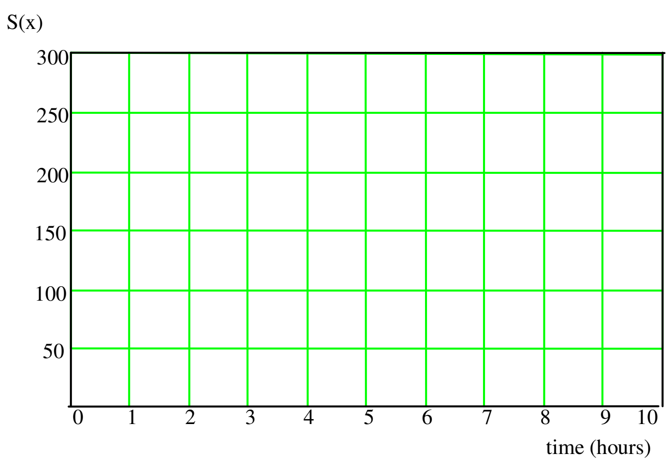
b. Since the definite integral represents the area under the curve, at any point, say $x = 3$ we can say that

$s(3) =$

c. We can consider this as a function then whose values represent the distance traveled, hence

$s(x) =$

d. In calculating these values, we are forming a new function that represents the accumulated area under the original function. Plot the values for this new function on a grid.

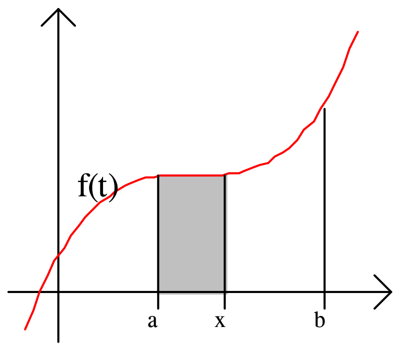


Def. **Accumulation Function**: Let f be continuous on $[a, b]$ and $x \in [a,b]$. The function

$A(x) = \int_a^x f(t)dt$ is called the accumulation function.

As x moves from a to b, the functional values represent the accumulation of signed area from a to x.

In the definition, if *f(t)* is a velocity function, then we know that *A(x)* would represent the distance traveled from time a to time x.



25. Suppose we start with the function $f(t) = 2t$. We want to calculate values for the accumulation function $A(x) = \int_0^x 2t\,dt$.

  a. Run ACCUMLAT to calculate values for *A(x)* over the set of points $x \in \{0, 0.2, 0.4, 0.6, \cdots, 4\}$.

  ACCUMLAT will ask for the starting point 0, the increment 0.2, and the number of points 20. When the program runs, it will store the x values in List 1 and the *A(x)* values in List 2. Finally it will graph *f(t)* and superimpose a scatter plot of *A(x)*.

  b. Examine the scatter plot and the values in List 1 and List 2.

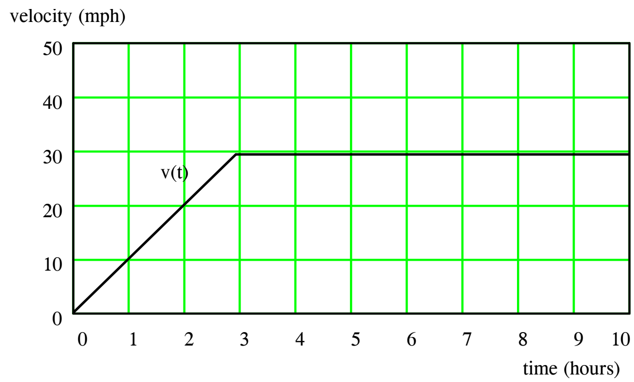
  c. Identify the function *A(x)*.

  d. What is the relationship between $A(x) = x^2$ and $f(t) = 2t$?

  e. How would $A(x)$ change if we changed the starting point a?

26. Let $f(t) = \frac{1}{t}$. Run ACCUMLAT and guess the accumulation function from looking at its graph.

27. Suppose we have a car whose velocity is graphed at right.



a. Write an integral expression to define $s(x)$.

b. Use the graph and grid to estimate values for the function.

$s(0) \approx$

$s(1) \approx$

$s(2) \approx$

$s(3) \approx$

$s(4) \approx$

$s(5) \approx$

$s(6) \approx$

$s(7) \approx$

$s(8) \approx$

$s(9) \approx$

c. Use the data values to try to write a formula for $s(x)$

$s(x) = \{$

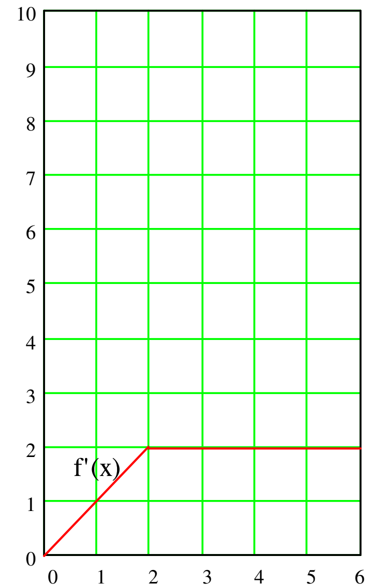
d. Write a formula for $v(t)$:

$v(t) = \{$

e. What is the relationship between $v(t)$ and $s(x)$?

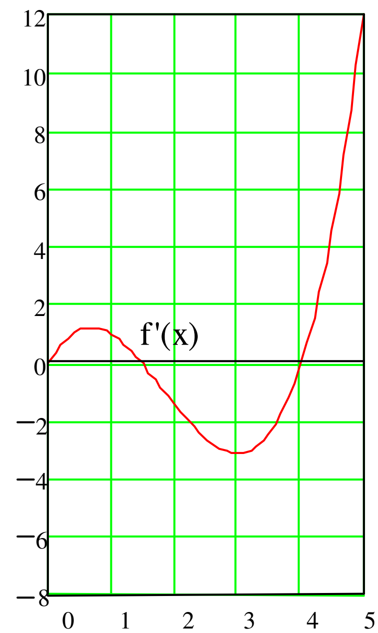
Constructing Antiderivatives Graphically Using Area

28. Use the graph at right of *f′(x)* and the Second Fundamental Theorem to sketch a graph of *f(x)* where $f(0) = 2$.



29. Do it again for $f(0) = -1$. How do the two antiderivatives differ?

30. Use the graph at right of *f′(x)* and the Second Fundamental Theorem to sketch a graph of *f(x)* where $f(0) = 1$.



Equations of Motion

31. A ball is thrown upward from the ground with a launching velocity of 96 ft/sec. What is the height of the ball after 3 seconds? How high does it go? When does it hit the ground?

32. A man driving an automobile in a straight line at a speed of 80 ft/sec applies the brakes at a certain instant. If the brakes furnish a constant acceleration of -20 ft/sec^2, how far will he go before he stops?

33. A ball is dropped from a height of 8 ft. When it hits the ground, it bounces back up with a speed that is three-fourths of its speed of impact. How high does it go after the first bounce?