

GUIDELINES FOR FINDING ABSOLUTE EXTREMA

To find the absolute maximum and/or minimum of a function, find each of the following:

- **Critical points:**
 - a. Find all critical points.
 - b. Calculate the functional value of each critical point.

- **Points of discontinuity:**
 - a. Find all points of discontinuity.
 - b. Find the left- and right-hand limit of the functional values at each point of discontinuity.

- **Endpoints:**
 - a. If there are endpoints, calculate the functional value at each endpoint.
 - b. If there are no endpoints, find the limit of the functional values on the left and right “ends” of the domain.

Compare the results of each of the above and choose the maximum or minimum value.

ROLLE'S THEOREM

Thm. Rolle's Theorem: If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Proof: Let $f(a) = f(b) = d$

Case 1. $f(x) = d$ for all x in (a, b) .

Why is the result obvious?

Case 2. $f(x) > d$ for some x in (a, b) .

See book.

Case 3. $f(x) < d$ for some x in (a, b) .

The function $f(x)$ must take on a minimum value at some point c in $[a, b]$.

Why?

Now $c \neq a$ and $c \neq b$, since $f(a) = f(b) = d$ and $f(c) < d$. So $c \in (a, b)$.

Since $f'(c)$ exists and $f(c)$ is a minimum, then $f'(c) = 0$.

Why?

-
- Determine if Rolle's Theorem can be applied to the given interval. If it can, then find all values of c in the interval such that $f'(c) = 0$.
 - $y = 3 - |x - 3|$, $[1, 5]$
 - $y = x^3 - 4x$, $[-2, 2]$
 - If a polynomial function has zeros at $\{-5, -2, 0, 6\}$, then what can be said about its relative extrema?

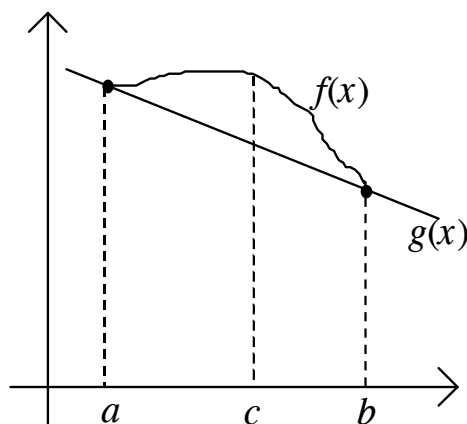
MEAN VALUE THEOREM

Thm. ❖ **Mean Value Theorem:** If f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof:

Let $g(x)$ be the line joining $(a, f(a))$ and $(b, f(b))$.

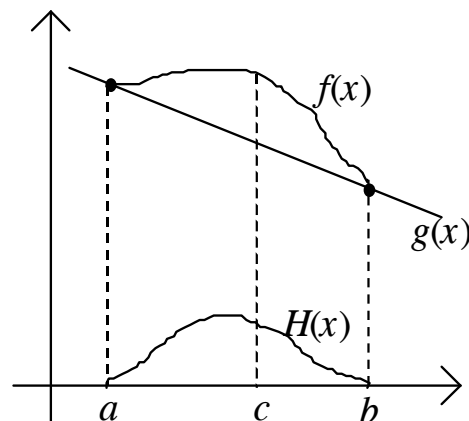
Then the equation of line $g(x)$ is:



Now construct a new function $H(x) = f(x) - g(x)$.

Show that $H(x)$ satisfies Rolle's Theorem.

a. $H(x)$ is continuous on $[a, b]$.



b. $H(x)$ is differentiable on (a, b) .

c. $H(a) = 0$ and $H(b) = 0$.

Hence by Rolle's Theorem, there exists a point c in (a, b) such that $H'(c) = 0$.

Find $H'(x)$ and use it to find $H'(c)$.

Since $H'(c) = 0$, then

Significance of the Mean Value Theorem

- a. *Geometric Interpretation:* Under the given conditions, there is a point in the open interval where the tangent to the curve is the same as the slope of the line joining the endpoints.
- b. *Application:* Under the given conditions, there is a point in the open interval where the instantaneous rate of change is the same as the average rate of change on the interval. If the function is a position function, then there is a point in the open interval where the instantaneous velocity is the same as the average velocity on the interval.

Applying the Mean Value Theorem

3. Find all values for c specified by the Mean Value Theorem.

a. $y = x(x^2 - x - 2)$ on $[-1, 1]$

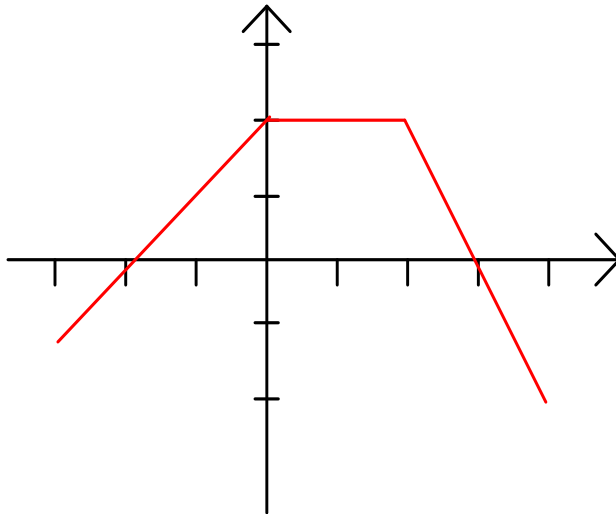
b. $y = \sqrt{x-2}$ on $[2, 6]$

INTERPRETING GRAPHS OF DERIVATIVES

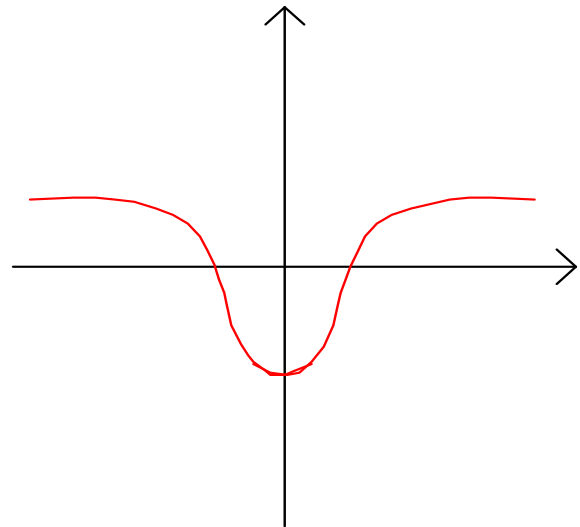
For each graph below, assume that it is a graph of $f'(x)$, where $f(x)$ is defined and continuous everywhere. Do the following for each:

- c. List all the things you know about $f(x)$ by looking at the given graph.
- d. Sketch a possible graph for $f(x)$.

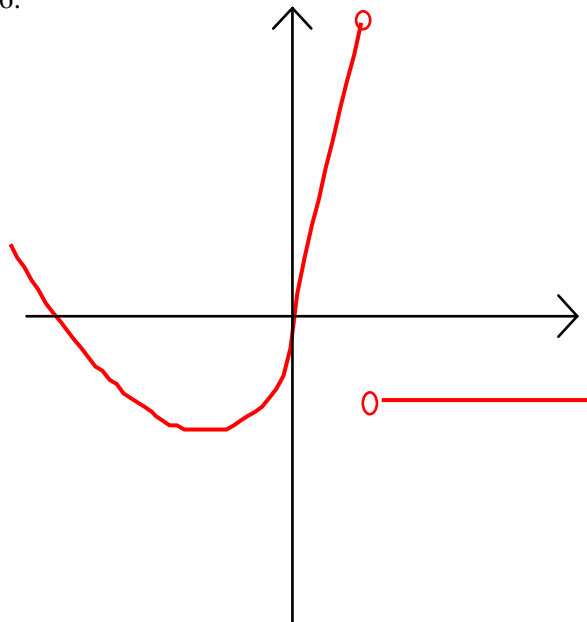
4.



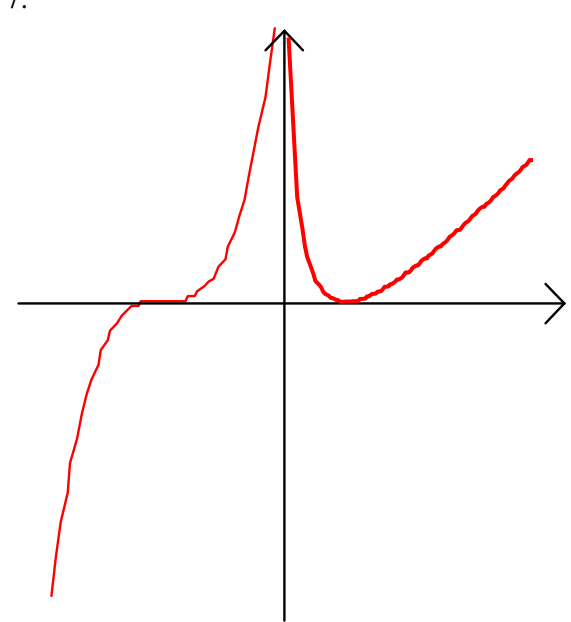
5.



6.



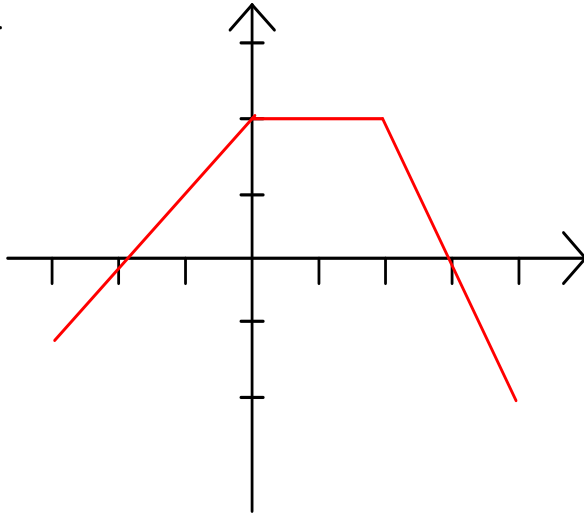
7.



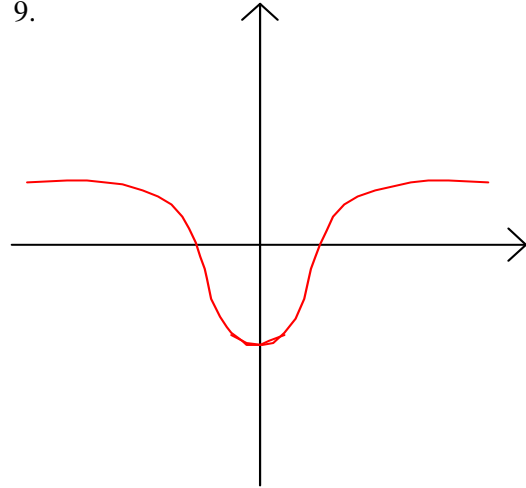
For each graph below, assume that it is a graph of $f''(x)$, where $f(x)$ is defined everywhere. Do the following for each:

- List all the things you know about $f(x)$ by looking at the given graph.
- Sketch a possible graph for $f(x)$.

8.



9.



12. A rectangular box, open at the top, with a square base, is to have a volume of 4000 cubic inches. What must be its dimensions if the box is to require the least possible material?

13. Find the coordinates of the point or points on the curve $x^2 - y^2 = 16$ which are nearest to the point $(0, 6)$.

OPTIMIZATION PROBLEMS – II

14. At noon two airplanes are headed towards the same city. One plane is 450 miles north of the city. It is flying due south at 450 mph. The other plane is 800 miles east of the city. It is flying at 600 mph due west. At what time will the planes be closest together?
15. A man is in a boat 4 miles off a straight coast. He wants to get home as quickly as possible. His house is 6 miles down the coast and 1 mile inland. If he can row 3 miles per hour and run 5 miles per hour, where should he land the boat?

16. Find the dimensions of the trapezoid of greatest area inscribed in $y = 16 - x^2$ and having its longer base on the x -axis?

17. A sphere has radius 9. Find the volume of the largest cone that can be inscribed in the sphere.

OPTIMIZATION PROBLEMS – III

18. Find the length of the longest pipe that can be carried around a right-angle corner if the two intersecting hallways are of width 5 feet and 7 feet.

19. The efficiency of a screw is given by the formula: $E = \frac{\tan \phi(1 - \mu \tan \phi)}{\mu + \tan \phi}$ where μ is the coefficient of friction and ϕ is the pitch angle of the screw. Find the angle ϕ for which the efficiency of the screw is a maximum. (Note: μ is a constant.)

20. A cone with a flat top is to be constructed out of two different metals. The metal used on the top costs 50% more than the metal used for the sides. If the cone must hold a total of 20 ft^3 , find the radius of the cone that will minimize the cost of the materials used in making the cone.

DIFFERENTIALS

- Def.* **Differential of y :** Let $y = f(x)$ represent a differentiable function and let dx be any nonzero change in x . Then the differential of y , written dy is given by $dy = f'(x)dx$.
- Def.* **Linear Approximation of Δy :** dy is called the linear approximation of the actual increment, Δy .
- Def.* **Linear Approximation of $f(x)$:** The expression $f(x) \approx f'(a)(x - a) + f(a)$ is called the linear approximation to $f(x)$ near $x = a$.
21. For $f(x) = x^4$ and $x = 3$, find the approximation for the change in y when x changes by 0.1.
22. Use differentials to approximate the value of $\sqrt[3]{124}$.
23. An object is dropped from the top of a 200 ft. building and its velocity function is $v(t) = -32t$.
- Approximately how far does it fall from $t = 1$ to $t = 1.5$?
 - Suppose we wanted the distance it fell during the first half second. Why would the differential produce a poor approximation? Try it. How could we change our work to get a meaningful approximation?

24. A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5° . How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

25. Write the linear approximation to the curve $f(x) = 3x^2 + 4$ at $x = -2$.

26. Given an unknown function $f(x)$ with values listed in the table below. Find an approximation for $f(16)$:

X	4	5	6	7	8	9	10	11	12	13	14
$f(x)$	21	21.1	21.2	21.5	22	22.6	23.4	24.2	24.8	25.3	25.6

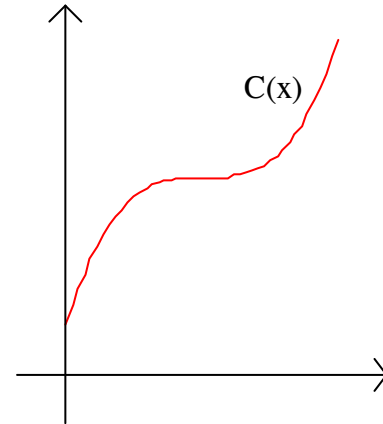
- a. First do it informally, just using common sense.
- b. Compare this to the linear approximation formula defined above:
 $f(16) \approx f'(14) \cdot (16 - 14) + f(14)$.

APPLICATIONS TO ECONOMICS

The Cost Function

Def. **Cost Function:** The cost function $C(x)$ represents the total cost of producing x number of units of some item.

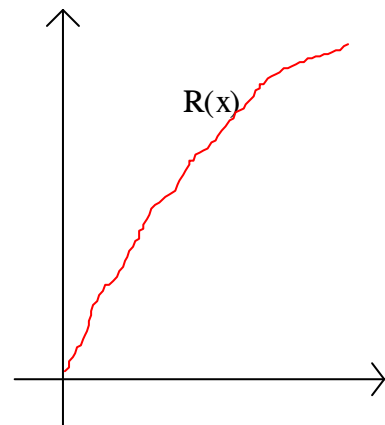
27. Explain the general shape of a cost function.
28. What does the slope of a *secant* line drawn from the origin represent?
29. What does the slope of a *tangent* line to the cost function represent?
30. What does the concavity of the cost function indicate?



The Revenue Function

Def. **Revenue Function:** The revenue function $R(x)$ represents the total money taken in when selling x number of units of some item.

31. Explain the general shape of a revenue function.
32. What does the slope of a *secant* line drawn from the origin represent?

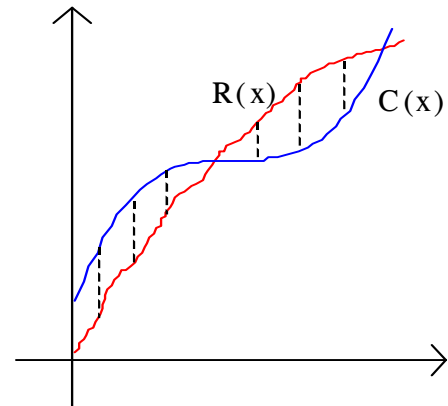


33. What does the slope of a *tangent* line to the revenue function represent?
34. What does the concavity of the revenue function indicate?

The Profit Function

Def. Profit Function: The profit function $P(x)$ is the difference between the revenue function and the cost function.

35. Where on the graph do you look to find the greatest profit?
36. How do we calculate the point where the maximum profit occurs?
- a. What is the relationship between $R'(x)$ and $C'(x)$ when profit is *increasing*?



- b. What is the relationship between $R'(x)$ and $C'(x)$ when profit is *decreasing*?
- c. What is the relationship between $R'(x)$ and $C'(x)$ when profit is at a relative maximum?

Marginal Analysis

In a decision as to whether or not to increase production, a company must compare the increased costs with the increased revenues. Suppose we focus on a single item.

37. How much more will it cost us if we make 1 more item?

38. How much more revenue will we take in if we make 1 more item?

Def. **Marginal Cost:** The derivative of the cost function is called the marginal cost and represents the additional cost of producing 1 more item.

Def. **Marginal Revenue:** The derivative of the revenue function is called the marginal revenue and represents the additional money taken in upon selling 1 more item.

39. What would you expect the term *marginal profit* to mean? How would it relate to marginal cost and marginal revenue?

40. What would the marginal profit be when the profit is at a relative maximum?

41. Given a cost function for a certain commodity of $C(x) = 0.02x^2 + 16x + 4000$. Find the marginal cost when production is at 100 units. Explain what that represents.

42. Suppose a yogurt firm finds that its revenue and cost functions are given by $R(x) = 15x^{\frac{1}{2}} - x^{\frac{3}{2}}$ and $C(x) = 3x^{\frac{1}{2}} + 4$, respectively, for $0.5 \leq x \leq 5$. Here x is measured in thousands of gallons, and $R(x)$ and $C(x)$ are measured in hundreds of dollars. Find the value for x that maximizes the profit.
43. A real estate office handles 50 apartment units. When the rent is \$180 per month, all units are occupied. However, on the average, for each \$10 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$12 per month for service and repairs. What rent should be charged to realize the most profit?