AVERAGE RATES OF CHANGE AND DIFFERENCE QUOTIENTS

Average Velocity – Numerically

1. Consider a swimmer swimming a 100 meter race in a 50 meter pool. Let $s(t)$ be a function that measures her distance from the starting wall as a function of time. Values for $s(t)$ are given in the table below.

t	0	5	10	15	20	25	30	35	40	45	50	55	55.5
$s(t)$	0	16.4	29.5	39.4	46.1	49.5	49.7	46.6	40.3	30.8	18.0	2.0	0

2. Understanding the data:

   a. Why do the values for *s(t)* increase and then decrease?

   b. At approximately what time does she turn around?

   c. What is her finishing time?

Def. **Average Velocity**: The average velocity of an object over an interval of time is the net change in position during the interval divided by the change in time. For a function *s(t)*, that is
$$\bar{v} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

3. Compute the average velocity on the interval of time [10, 25].

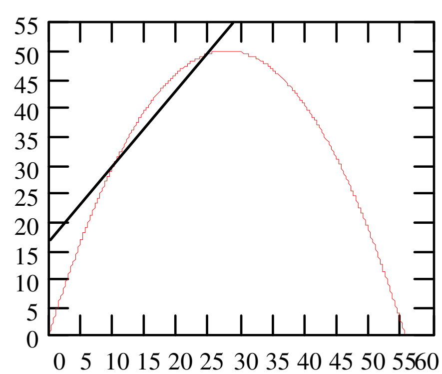
4. What is her average velocity for the entire race?

Average Velocity – Graphically

5. *Average velocity is the slope of the line* which passes through the two points $(t_1, s(t_1))$ and $(t_2, s(t_2))$.

6. Such a line is called a **secant line** because it intersects the graph in more than one point.

7. Use the graph and a secant line to estimate average velocity on [10, 25].



8. By looking at the graph, name several 5-second intervals of time over which her average velocity would be negative.

9. By looking at the graph, name several intervals of time over which her average velocity would be zero.

Difference Quotients – Analytically

10. We want to generalize the work with velocities to finding an average rate of change of any function. We call this expression a difference quotient.

Def. ❖**Difference Quotient**: The expression $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ is called a difference quotient and represents the average rate of change of *f(x)* over the interval $[x_1, x_2]$.

11. Graphically, the difference quotient is the slope of a secant line.

12. Suppose $f(t)=\dfrac{700}{t^2+4t+10}$ is a function which measures the temperature of food put in a freezer with respect to time t. Then the expression $\dfrac{f(t_2)-f(t_1)}{t_2-t_1}$ represents the average rate of change of the temperature of the food over the time interval $[t_1, t_2]$.

13. Find the average rate of change in the temperature during the first 2 hours.

14. Alternate notation for a difference quotient:

    a. $\dfrac{f(x+h)-f(x)}{h}$ or

    b. $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

15. Notice that in each case the denominator is the difference between the *x*-coordinates of the two points at which the function is evaluated in the numerator.

16. Use TI-89 with function notation to find the average rate of change in the temperature from 2 hours to 6 hours.

F4 *Enter* $f(x)=700/\left(x^2+4x+10\right)$ *Enter* $\left(f(6)-f(2)\right)/(6-2)$ *Enter*

INSTANTANEOUS RATES OF CHANGE AND THE DERIVATIVE

Instantaneous Velocity

17. We take the limit of the values for the average velocity, as the time interval becomes increasingly small, to get an *instantaneous* velocity.

Def. **Instantaneous Velocity**: The instantaneous velocity of an object at time t_1 is given by the limit of the average velocity as t_2 approaches t_1. For the function *s(t),* that is

$$v(t_1) = \lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

18. Graphically, the instantaneous velocity corresponds to the slope of the tangent line.

Derivative as an Instantaneous Rate of Change

19. In the previous lesson we generalized the process of calculating an average velocity to finding any average rate of change, which we called the difference quotient, expressed as $\frac{f(t_2) - f(t_1)}{t_2 - t_1}$.

20. Now we are going to generalize the process of finding an instantaneous velocity to finding any *instantaneous* rate of change by taking the limit of the difference quotients as the interval between the *x*-values becomes increasingly small. We call this limit the derivative.

Def. ❖**Derivative**: $f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ (if it exists) is called the derivative of *f(x)* at x_1 and represents the instantaneous rate of change of *f(x)* at the point x_1.

Def. ❖**Derivative, Alternate Forms**: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (if it exists) or

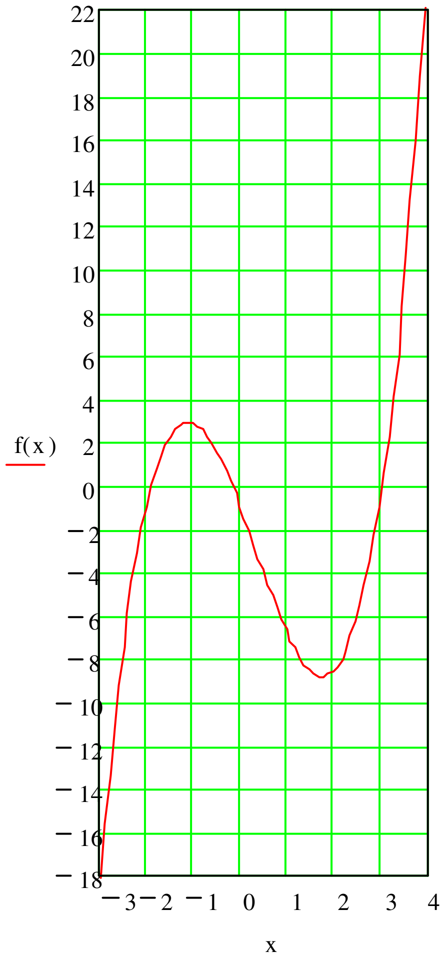
$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ (if it exists).

Derivative – Graphically

21. Graphically, the derivative is the slope of the tangent line.

Def. **Tangent Line**: If $f(x)$ is defined on an open interval containing c, and if the derivative $f'(c)$ exists, then the line passing through $(c, f(c))$ with slope $f'(c)$ is the tangent line to the graph of $f(x)$ at the point $(c, f(c))$.

22. Consider the function graphed at right.



a. Use the graph to estimate the slope of the tangent line at every integer point on the curve. Write the values in the table below.

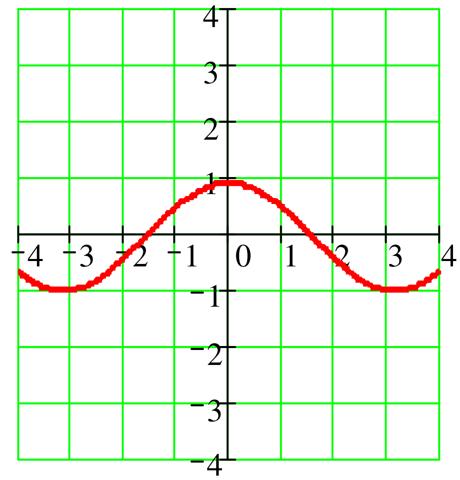
x	-3	-2	-1	0	1	2	3	4
$f'(x)$								

b. The values obtained for the slope of the tangent line define a new function which we call the derivative function. Since it is like any other function we can draw its graph. Do so by drawing over top of the given graph.

c. What shape does the derivative have? What type of function does it appear to be?

23. Consider the function graphed at right.

 a. Use the graph to estimate the slope of the tangent line at every integer point on the curve. Write the values in the table below.



x	-3	-2	-1	0	1	2	3	4
$f'(x)$								

 b. Plot the values of the derivative and sketch in its graph.

 c. What shape does the derivative have? What type of function does it appear to be?

FINDING THE DERIVATIVE FUNCTION

Approximating the Derivative – Numerically

24. Recall the example from a previous lesson (two lessons back) of placing food in the freezer, where the temperature of the food at any time t is found by $f(t)=\dfrac{700}{t^2+4t+10}$.

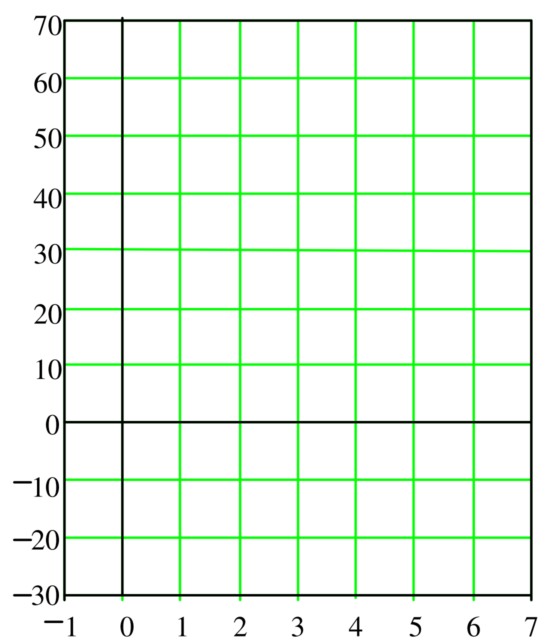
25. The *average* rate of change of the temperature of the food placed in the freezer at time t_1 was $\dfrac{f(t_2)-f(t_1)}{t_2-t_1}$.

26. The *instantaneous* rate of change of the temperature of the food placed in the freezer at time t_1 would be $\lim\limits_{t_2\to t_1}\dfrac{f(t_2)-f(t_1)}{t_2-t_1}$.

27. Find approximations for the derivative of the temperature function using the table below.

t	0	1	2	3	4	5	6
$f(t)$	70	46.7	31.8	22.6	16.7	12.7	10
$f'(t)$							

28. Plot the function (by copying a TI-89 graph) and its derivative on the axes at right.



Finding the Derivative – Analytically

29. Recall the definition of the derivative as $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

30. Given $f(x) = 3x^2 - 5x$.

a. Find $f'(x)$.

b. Find the slope of the tangent line to the curve at $x = 0$.

c. Find the instantaneous rate of change of y with respect to x at $x = 1$.

31. Given $f(x) = -x^2 + 2x - 4$.

a. Find $f'(x)$.

b. Find the point where the slope of the tangent line is zero.

Finding the Derivative – Analytically, Assisted by the TI-89

32. Given $f(x) = 2x^3 - 6$.

    a. Find $f'(x)$ assisted by the TI-89.
    $\left(\left(2(x+h)^3 - 6\right) - \left(2x^3 - 6\right)\right) \div h$, Enter

    b. Find the slope of the tangent line to the curve at $x = 2$.

33. Given $y = \dfrac{1}{3x}$.

    a. Find $f'(x)$ assisted by the TI-89.
    $\left(1 \div \left(3(x+h)\right) - 1 \div (3x)\right) \div h$, Enter, F2, ComDenom

    b. Find the point on the curve where the slope of the tangent line is equal to -4.

DIFFERENTIABILITY

Differentiability and Local Linearity

Def. **Differentiability**: A function is said to be differentiable at a point if it has a derivative at the point.

34. Can a curve be locally linear and not differentiable?

35. Graph the function $y = (x-1)^{\frac{1}{3}} + 1$ using a window size of $0 \le x \le 2$ and $0 \le y \le 2$.

    a. *ZoomIn* on the point $(1, 1)$ three times.

    b. Describe the tangent line.

    c. Would the function have a derivative at that point?

Def. **Vertical Tangent Line**: If $f(x)$ is continuous at $x = c$ and $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = +\infty$ (or $-\infty$), then the line $x = c$ is called the vertical tangent to the curve at $(c, f(c))$.

Thm. **Local Linearity and Differentiability**: If a curve is locally linear at a point $x = c$ and the tangent line is *not* vertical there, then the function is differentiable at $x = c$.

Differentiability and Continuity

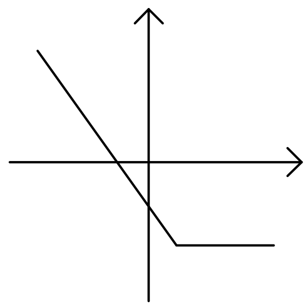
36. Can a curve be continuous and not differentiable?

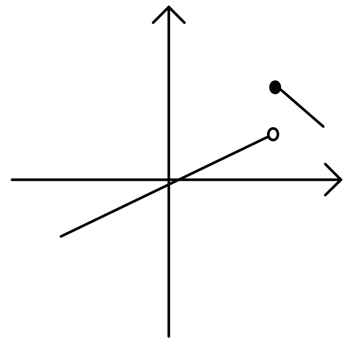
Thm. **Continuity and Differentiability**: If a curve is differentiable at a point $x = c$, then it is continuous at $x = c$.

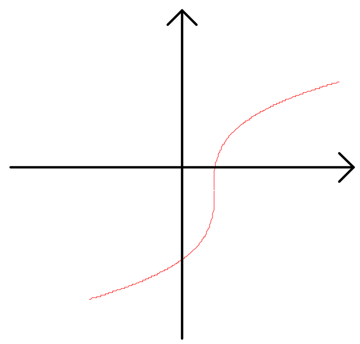
Proof:

Determining Differentiability Graphically

37. Where is the function not differentiable?



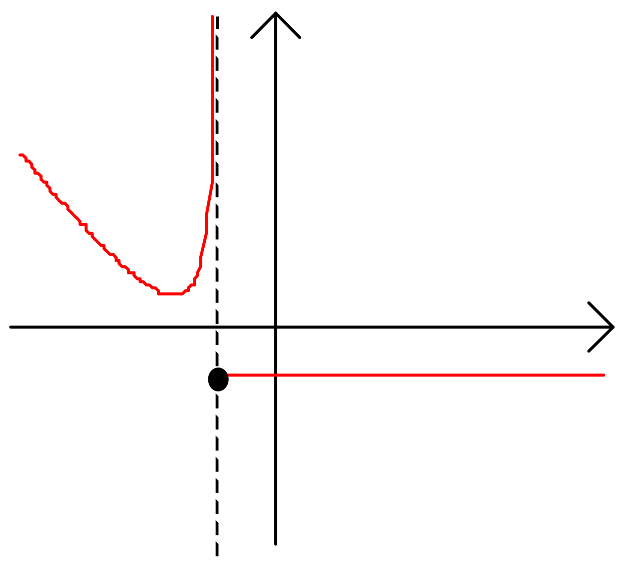




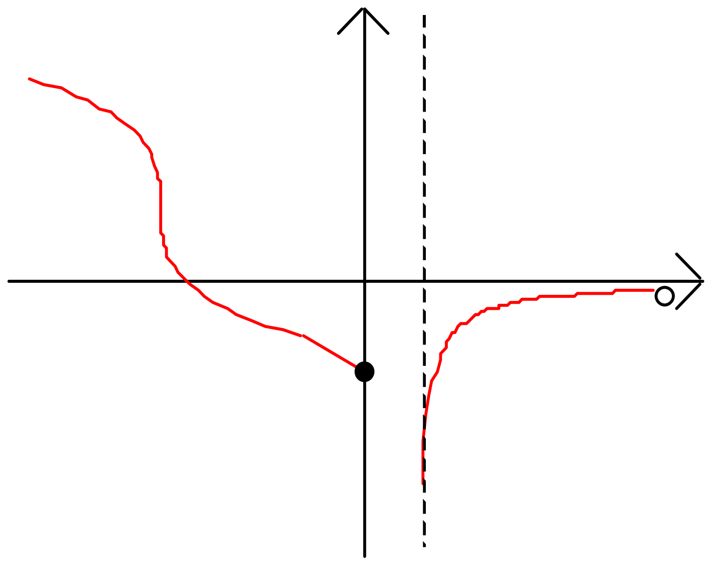
38. The three conditions that make a function not differentiable at a point are:

Sketching Derivatives

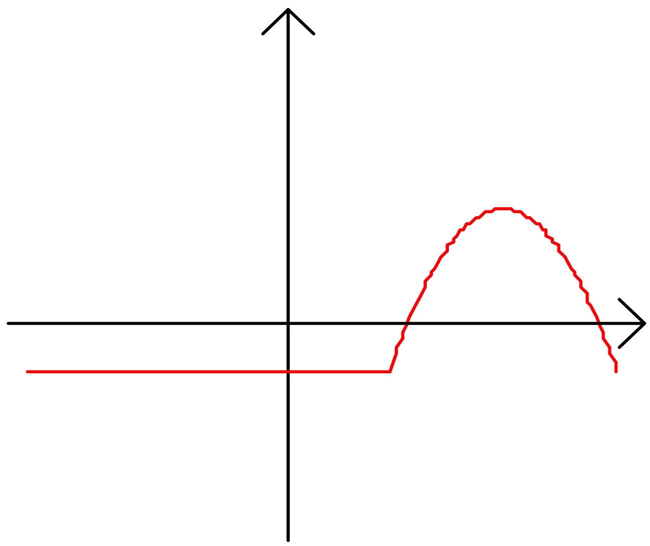
39. Draw the derivative.



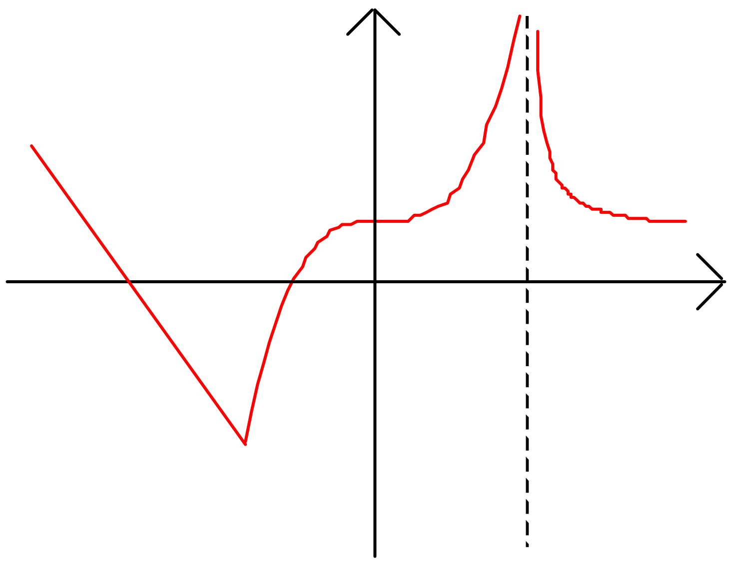
40. Draw the derivative.



41. Draw the derivative.



42. Draw the derivative.



DERIVATIVE PROPERTIES

Properties of Derivatives

Thm. **Properties of Derivatives**:

a. If $y = c$, then $y' = 0$.

b. If $y = c \bullet f(x)$, then $y' = c \bullet f'(x)$.

Proof:

$$y' = \lim_{h \to 0} \frac{\rule{4cm}{0.4pt}}{h}$$ Definition of the Derivative

$$= \lim_{h \to 0} \frac{c \bullet [\qquad\qquad\qquad]}{h}$$ Distributive Property

$$= c \bullet \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ ______________________

$$= c \bullet$$ ______________________

c. If $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$.

Proof:

$$y' = \lim_{h \to 0} \frac{\rule{4cm}{0.4pt}}{h}$$ Definition of the derivative

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$ Distributive and Commutative Prop

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$ Addition of Fractions

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$ ______________________

$$=$$ ______________________

d. If $y = x^n$, then $y' = nx^{n-1}$.

Differentiating the Sine and Cosine

Thm. **Derivative of the Sine**: If $y = \sin x$, then $y' = \cos x$.

Proof:

$$y' = \lim_{h \to 0} \frac{\qquad\qquad\qquad\qquad}{h}$$ Definition of the Derivative

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$ Trig Addition Formula

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$ Commutative Property

$$= \lim_{h \to 0} \frac{\sin x[\qquad\qquad\qquad] + \cos x \sin h}{h}$$ Distributive Property

$$= \lim_{h \to 0} \left[\frac{\sin x[\cos h - 1]}{h} + \frac{\cos x \sin h}{h} \right]$$ Addition of Fractions

$$= \lim_{h \to 0} \left[\frac{\sin x[\cos h - 1]}{h} \right] + \lim_{h \to 0} \left[\frac{\cos x \sin h}{h} \right]$$ ______________________

$$= (\sin x) \bullet \lim_{h \to 0} \frac{[\cos h - 1]}{h} + (\cos x) \bullet \lim_{h \to 0} \frac{\sin h}{h}$$ ______________________

$$= (\sin x) \bullet [\quad] + (\cos x) \bullet [\quad]$$ Trig Limit Theorems

$$= \cos x$$ Simplification

Thm. **Derivative of the Cosine**: If $y = \cos x$, then $y' = -\sin x$.

INTERPRETING THE DERIVATIVE

43. Suppose that a balloon is being blown up and the volume (in cubic inches) of air in the balloon at time t (in seconds) is described by the function $V(t)$.

    a. What is the meaning of $V'(t)$ and what would be its units of measure?

    b. Sketch a graph of $V'(t)$, assuming that the balloon is being blown up by a person, not a machine.

44. Suppose that the function $P(x)$ represents the profit in dollars of manufacturing x items.

    a. What is the meaning of $P'(x)$ and what would be its unit of measure?

    b. What is the meaning of $P'(50) = 15.60$?

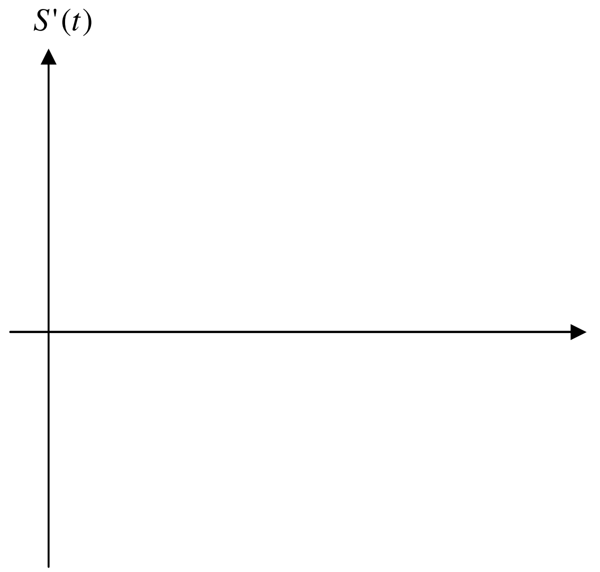
    c. What is the meaning of $P'(100) = -2.25$? Explain in practical terms how this could happen.

    d. Of what interest would a graph of $P'(x)$ be to managers in the corporation manufacturing these items?

45. Suppose that $S(t)$ is a function representing the position of a car measured in feet from some starting point, with respect to time measured in seconds.

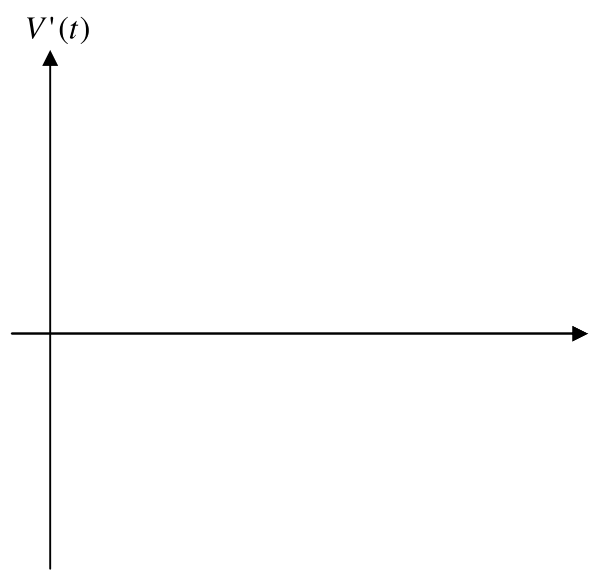
a. What is the meaning of $S'(t)$ and what would be its unit of measure?

b. Suppose that this car is driving along the number line so that when it is positioned to the right of its starting point, the distance is considered positive and when it is to the left of its starting point, its position is considered negative. The car pulls out slowly to the right, speeds up, then slows down and comes to a stop, turns around, pulls out rapidly to the left until he gets to a desired speed and then continues at that same speed. Draw a graph of $S'(t)$ from this description.



c. Suppose that $V(t)$ is a function representing the velocity (in feet per second) of the above car with respect to time t (in seconds). What is the meaning of $V'(t)$ and what would be its unit of measure?

d. Draw a graph of $V'(t)$ from the previous description of the car's motion.



46. An object is moving horizontally (with positive positions to the right) so that its position is found by the function $s(t) = t^3 - 3t^2 - 9t + 4$.

a. What does $s'(t)$ represent? Find $s'(t)$.

b. What does $s''(t)$, represent? Find $s''(t)$.

c. When is the object stopped?

d. When is it moving to the right? to the left?

e. Find the average velocity over the interval $[2, 4]$.

f. What is the instantaneous velocity at time $t = 3$?

g. Find the average acceleration over the interval $[2, 4]$.

h. Find the instantaneous acceleration at $t = 3$?

47. A ball is thrown upward from ground level with initial velocity 96 ft/sec. Its position equation is $s(t) = -16t^2 + 96t$.

    a. Find the velocity of the ball at any time t.

    b. Is the ball rising or falling at time $t = 2$?

    c. Is its velocity increasing or decreasing at that time? Why?

    d. Is its *speed* increasing or decreasing at that time?

    e. Is the ball rising or falling at time $t = 5$?

    f. Is its velocity increasing or decreasing at that time? Why?

    g. Is its *speed* increasing or decreasing at that time?

    h. What is the maximum height that the ball attains?

    i. With what velocity does it hit the ground?

PRODUCT AND QUOTIENT RULES

Thm. **Product Rule**: If *f(x)* and *g(x)* are differentiable functions at x, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x).$$

Proof:

$$\frac{d}{dx}\left[f(x)g(x)\right] =$$

$$\lim_{h\to 0}\frac{\qquad\qquad\qquad\qquad}{h} = \qquad \text{Definition of the Derivative}$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} = \qquad \text{Inverse Prop for Addition}$$

$$\lim_{h\to 0}\frac{[\qquad\qquad]g(x+h) + f(x)[\qquad\qquad]}{h} = \qquad \text{Distributive Property}$$

$$\lim_{h\to 0}\left[\frac{[f(x+h)-f(x)]}{h}g(x+h) + f(x)\frac{[g(x+h)-g(x)]}{h}\right] = \qquad \text{Properties of Fractions}$$

$$\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}g(x+h) + \lim_{h\to 0} f(x)\frac{[g(x+h)-g(x)]}{h} = \qquad \underline{\qquad\qquad\qquad}$$

$$\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h} \bullet \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \bullet \lim_{h\to 0}\frac{[g(x+h)-g(x)]}{h} = \qquad \underline{\qquad\qquad}$$

$$[\qquad] \bullet \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \bullet [\qquad] = \qquad \text{Definition of a Derivative}$$

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad} \qquad \text{Limit Prop of Continuous Function}$$

Thm. **Quotient Rule**: If *f(x)* and *g(x)* are differentiable functions at x, and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}.$$

Thm. **Derivative of the Tangent**: If $y = \tan x$, then $y' = \sec^2 x$.

Thm. **Derivative of the Cotangent**: If $y = \cot x$, then $y' = -\csc^2 x$.

Thm. **Derivative of the Secant**: If $y = \sec x$, then $y' = \sec x \tan x$.

Thm. **Derivative of the Cosecant**: If $y = \csc x$, then $y' = -\csc x \cot x$.

CHAIN RULE

Thm. ❖**Chain Rule**: If $y = f(g(x))$ is a differentiable function of $g(x)$, and $g(x)$ is a differentiable function of x, then $\frac{d}{dx}[f(g(x))] = f'(g(x)) \bullet g'(x)$.

Thm. ❖**Alternate Form of Chain Rule**: If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$.

Proof:

$$y' = \lim_{h\to 0} \frac{\qquad\qquad}{h}$$ Definition of the Derivative

$$= \lim_{h\to 0}\left[\frac{f(g(x+h)) - f(g(x))}{h} \bullet \frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right]$$ ________________

$$= \lim_{h\to 0}\left[\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \bullet \frac{g(x+h)-g(x)}{h}\right]$$ Properties of Fractions

$$= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \bullet \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$ ________________

$$= \lim_{\substack{h\to 0 \\ x+h\to x \\ g(x+h)\to g(x)}} \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \bullet \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$ Limit Prop of Cont Func

= ________________________ Definition of the Derivative

Thm. **Absolute Value Rule**: If $y = |x|$, then $y' = \frac{|x|}{x}$.

IMPLICIT DIFFERENTIATION

48. Explicit Function: $y = 3x^2 - 2$. Take the derivative of both sides with respect to x.

49. Implicit Function: $3x^2 - y = 2$. Again, take the derivative of both sides with respect to x.

50. Find $\frac{dy}{dx}$ in terms of x: $xy = 4$.

51. Where does the slope of $x^2 + y^2 = 9$ have the value -1?

52. Find $\frac{dy}{dx}$ at (0, 0) when $2x^2 + \sqrt{xy} = 2y$.

53. Use the TI-89 to determine where the curve $2x^2 - 3xy + 3y^2 = 10$ has a vertical tangent.
    d($2x\text{^}2 - 3x * y(x) + 3y(x)\text{^}2 = 10, x)$ ENTER

    Solve for $\frac{dy}{dx}$ by hand.

    Find the point that satisfies the original equation and makes $\frac{dy}{dx} = 0$.

    CUSTOM F3 Solve($6y - 3x = 0$ and $2x\text{^}2 - 3x * y + 3y\text{^}2 = 10$,{*x*,*y*}) ENTER
    (CAUTION! Put a multiplication sign between the *x* and *y* variables. If you still have problems, put parentheses around the two equations.)

54. Write the linear approximation to the curve $\sin x = \tan y$ at $(0, 0)$.

55. Find $\frac{d^2y}{dx^2}$ in terms of x and y where $x + y = \cos(x - y)$.

56. Suppose that an object is traveling on a coordinate system so that its *x*- and *y*-coordinates are each a function of time, expressed as *y(t)* and *x(t)* where $y = 3x^2 - 4x$. Differentiate implicitly with respect to *time* using the Chain Rule.

57. Differentiate with respect to t: $y = -x^3 + 5xy - 8$.

58. If $y = 4x^2 + x$ and $\frac{dy}{dt} = 3$, find $\frac{dx}{dt}$ when $x = 1$.

59. A point moves along the curve $y = \sqrt{x^2 + 1}$ in such a way that $\frac{dx}{dt} = 4$. Find $\frac{dy}{dt}$ when $x = 3$.

RELATED RATES – I

Suggested Procedure for Solving Related Rate Problems:

- Draw and label an appropriate figure.
- Write each rate (that is given or asked for) as a derivative.
- Write an equation that relates all of the variables involved in the above derivatives.
- Differentiate the equation with respect to time.
- Substitute each given rate and given quantity into the equation.
- Solve for the remaining rate.

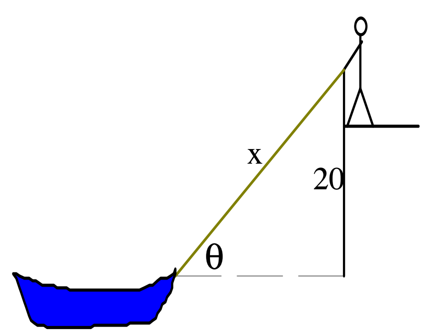
60. Water leaking onto a floor creates a circular pool with an area that increases at the rate of 3 square centimeters per minute. How fast is the radius of the pool increasing when the radius is 10 centimeters?

61. The base of a triangle is increasing at the rate of 3 in/min, while the altitude is decreasing at the same rate. At what rate is the area changing when the base is 10 inches and the altitude is 6 inches?

62. A car pulls away from a house, drives 0.25 miles, and makes a right angle turn. If it then proceeds at a speed of 35 mph, find the rate at which its distance from the house is changing when it is 0.5 mile from the turn.

63. A giraffe, 20 feet tall, walks along a football stadium towards one of the light poles. If the light pole is 50 feet high, and the giraffe is walking at the speed of 5 ft/sec, how fast is the shadow of the giraffe shrinking when he is 25 feet away from the pole?

64. A man on a wharf is pulling in a small boat. His hands are 20 feet above the level of the point on the boat where the rope is tied. If he is pulling in the rope at 2 feet per second, how fast is the angle that the rope makes with the horizontal increasing when there is 52 feet of rope out?



RELATED RATES II

65. The relationship between the escape velocity v of a star and its radius r (in kilometers) is given by $v^2 r = 2GM$, where $G = 6.67 \times 10^{-11}$ and M is the mass of the star in kilograms. Suppose that an old star with mass 4×10^{30} kilograms is in the process of collapsing and becoming an exceedingly dense neutron star. Assume that at the instant the star has radius 45,000 kilometers, its radius is decreasing at the rate of 3×10^{-6} kilometers per second. How fast is its escape velocity increasing at that instant?

66. A trough 10 feet long has as its ends isosceles trapezoids. The altitude is 2 feet, the lower base is 2 feet, and the upper base is 3 feet. If water is let in at the rate of 3 cubic feet per minute, how fast is the water level rising when the water is 1 foot deep?

67. Water is leaking out of a conical tank (vertex down) at the rate of 0.5 ft^3/min. The tank is 30 feet across at the top and 10 feet deep. If the water level is rising at the rate of 1.5 ft/min when the water level is 4 feet deep, at what rate is water being poured into the tank from the top?

68. At noon of a certain day, ship A is 60 miles north of ship B. if A sails east at 15 mph and B sails north at 12 mph, determine how rapidly the distance between them is changing 2 hours later.

69. 5. A lighthouse is 2 miles off a straight shore. Its light makes 3 revolutions per minute. How fast does the light beam move along a sea wall at a point 2 miles down the coast?